Degrees High for Isomorphisms

Wesley Calvert

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NERDS 17

Theorem

Let $\mathcal{V}$ be a computable vector space over $\mathbb{Q}$. Then

1. *If $dim(\mathcal{V}) < \aleph_0$, then for every computable $\mathcal{U} \cong \mathcal{V}$ there is a computable isomorphism $f : \mathcal{U} \to \mathcal{V}$.*

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2. *If $dim(\mathcal{V}) = \aleph_0$, then there is a computable $\mathcal{U} \cong \mathcal{V}$ such that no computable function $f : \mathcal{U} \to \mathcal{V}$ is an isomorphism.*

Definition

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1. *$\mathcal{V}$ is $\emptyset'$-categorical*
2. *$\mathcal{V}$ is $\emptyset$-categorical if and only if $\dim_{\mathbb{Q}} V < \aleph_0$.*

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3. *If* $\dim_{\mathbb{Q}} V = \aleph_0$, *then* $\dim_{\emptyset} V = \aleph_0$.

Let $\mathcal{K}$ be a class of computable structures, and $X \subseteq \mathbb{N}$.

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Question

For what X is it true that for every $\mathcal{A} \in \mathcal{K}$, the structure $\mathcal{A}$ is X-categorical?

Definition

Let $\mathcal{M}$ be a computable structure. The categoricity spectrum of $\mathcal{M}$ is the set

$$\{\mathbf{d} : \dim_{\mathbf{d}} \mathcal{M} = 1\}.$$

Definition

We say that $\mathcal{A}$ is relatively X-categorical if for any $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism $f : \mathcal{A} \to \mathcal{B}$ computable in $\mathcal{B} \oplus X$.

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Problem (Goncharov, et al.)

Let $\mathcal{A}$ be Δ^1_1-categorical. Must $\mathcal{A}$ be relatively Δ^1_1-categorical?

Definition

A degree $\mathbf{d}$ is said to be low for isomorphism if and only if for every pair of computable structures $\mathcal{A}, \mathcal{B}$,

- there is $f : \mathcal{A} \to \mathcal{B}$ with $f \leq_T \mathbf{d}$ if and only if there is some computable $g : \mathcal{A} \to \mathcal{B}$.

Theorem (Franklin–Solomon 2014)

There are degrees $\mathbf{d} \not\equiv_T \emptyset$ *which are low for isomorphism.*

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Every example of the following is low for isomorphism:

1. *Cohen 2-generic*
2. *Matthias 2-generic*

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There exist degrees low for isomorphism with the following combinations of properties:

1. *Hyperimmune free and minimal*
2. *Hyperimmune free and not computably traceable*

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Degrees with the following properties are not low for isomorphism:

1. *Non-computable Δ^0_2 degrees*
2. *Degrees computing separating sets for computably inseparable c.e. sets.*

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Theorem (Franklin–Solomon 2014)

The set of degrees which are low for isomorphism has measure 0.

Definition

We say that a degree $\mathbf{d}$ is the degree of categoricity of $\mathcal{M}$ if and only if it is the least degree in the categoricity spectrum of $\mathcal{M}$.

Theorem (Fokina–Kalimullin–Miller 2010, Csima–Franklin–Shore 2012)

If α is any computable ordinal and $\emptyset^{(\alpha)} \leq_T \mathbf{d}$ and $\mathbf{d}$ is d.c.e. in $\emptyset^{(\alpha)}$, then there is a computable structure $\mathcal{M}$ such that $\mathbf{d}$ is the degree of categoricity of $\mathcal{M}$

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Problem

Can this be extended to 3-c.e.?

Proposition

If $\mathbf{d} \gneq_T \emptyset$ *is a degree of categoricity, then it is not low for isomorphism.*

Definition

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2. $\mathbf{d}$ is said to be high if and only if $\mathbf{d}' \equiv_T (\emptyset')'$.

Definition

We say that a degree $\mathbf{d}$ is high for isomorphism if for any computable structures $\mathcal{A} \cong \mathcal{B}$, there is $f : \mathcal{A} \cong \mathcal{B}$ with $f \leq_T \mathbf{d}$.

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Question

Could such a thing even exist?

Theorem (Scott)

Let $\mathcal{M}$ be a countable structure. Then there is a sentence $\varphi_{\mathcal{M}}$ of $L_{\omega_1\omega}$ such that for any countable structure $\mathcal{N}$, if $\mathcal{N} \models \varphi_{\mathcal{M}}$, then $\mathcal{N} \cong \mathcal{M}$.

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Theorem (Harrison-Trainor 2018)

There is.

Definition

We define the back-and-forth relations on pairs of the form $(\mathcal{A}, \bar{a})$, where $\mathcal{A}$ is a structure and $\bar{a}$ is a finite tuple of elements of $\mathcal{A}$:

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2. For $\alpha \geq 1$, we define $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if and only if for each $\bar{d}$ in $\mathcal{B}$ and each $\beta < \alpha$, there is $\bar{c} \subseteq \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

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We also define $(\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})$ if and only if both $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ and $(\mathcal{B}, \bar{b}) \leq_\alpha (\mathcal{A}, \bar{a})$.

Example

Let $\mathcal{H}_1, \mathcal{H}_2$ be linear orderings of type $\omega_1^{CK}(1+\mathbb{Q})$.

Theorem (Nadel 1974)

For any computable structures $\mathcal{A}, \mathcal{B}$, *if for all* $\alpha < \omega_1^{CK} + 1$ *we have* $\mathcal{A} \equiv_\alpha \mathcal{B}$, *then* $\mathcal{A} \cong \mathcal{B}$.

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Corollary

$\mathcal{O}'^{?}$ is high for isomorphism.

Definition

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Example

Let $\mathcal{M}_1, \mathcal{M}_2$ be computable structures. Then the class of isomorphisms $f : \mathcal{M}_1 \to \mathcal{M}_2$ is a Σ^1_1 class.

Theorem (Gandy)

Let A be a Σ^1_1 class. Then there is $f \in A$ such that $\omega_1^f = \omega_1^{CK}$.

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Corollary

$\mathcal{O}$ is high for isomorphism.

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We say that a degree **d** is uniformly high for isomorphism if and only if there is a single algorithm that will, given a pair of isomorphic computable structures, compute from **d** an isomorphism.

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Proposition

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If $\mathbf{d}$ *is high for isomorphism, then* $\mathbf{d}''$ *is uniformly high for isomorphism.*

Proposition (C.-Franklin-Turetsky)

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2. *If* $\mathbf{d}$ *is uniformly high for isomorphism, then* $\mathbf{d}'$ *enumerates* $\mathcal{O}$.

Theorem (C.-Franklin-Turetsky)

There is a degree $\mathbf{d}$ *which is high for isomorphism with* $\mathbf{d}''' \equiv_T \mathcal{O}$.

Proof.

A <u>pointed perfect tree</u> is an increasing funciton $f : 2^{<\omega} \rightarrow 2^{<\omega}$ such that for every path X, we have $f \leq_T f(X)$.

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We build an $\mathcal{O}$ computable refining sequence of uniformly pointed perfect trees.

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In even steps, we work to force that $'''$ is $\mathcal{O}$.

In odd steps, we work to arrange that this sequence of functions approaches a high for isomorhipsm. □

Corollary

There is a degree $\mathbf{d}$ *which is uniformly high for isomorphism with* $\mathbf{d}' \equiv_T \mathcal{O}$.

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2. For $\alpha \in \mathcal{H}$ with $\alpha > 0$, we define $(\mathcal{A}, \bar{a}) \preceq_\alpha (\mathcal{B}, \bar{b})$ if and only if for each $\bar{d}$ in $\mathcal{B}$ and each $\beta < \alpha$, there is $\bar{c} \subseteq \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \preceq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

Definition

Given a Harrison ordering $\mathcal{H}$, a jump structure on $\mathcal{H}$ is a system $\{(x, S_x) : x \in \mathcal{H}\}$ such that for all x, we have $S_x \subseteq \mathbb{N}$ and if $y <_{\mathcal{H}} x$, we have $S'_y \leq_T S_x$.

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Definition

We say that a degree $\mathbf{d}$ is uniformly jump complete if and only if it will compute, uniformly in a Harrison ordering $\mathcal{H}$, a jump structure on $\mathcal{H}$.

Proposition (C.-Franklin-Turetsky)

$UHFI \to UJC \to USC$

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Part of Proof.

Suppose **d** is UJC.

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UHFI $\rightarrow$ *UJC* $\rightarrow$ *USC*

Part of Proof.

Suppose **d** is UJC.
A single point in the non-well-founded part of a Harrison ordering computes all Δ_α relations. □

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There is a uniformly Scott complete degree which is low for ω_1^{CK}.
If $\mathbf{d}$ is uniformly jump complete, then $\mathbf{d}'' \geq \mathcal{O}$. □

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There is a uniformly jump complete degree $\mathbf{d}$ *with* $\mathbf{d}'' \equiv_T \mathcal{O}$.

Definition

We say that a degree **d** is uniformly high for descending sequences if and only if it will compute, uniformly in a Harrison ordering $\mathcal{H}$, a descending sequence in $\mathcal{H}$.

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Proposition (C.-Franklin-Turetsky)

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Theorem (C.-Franklin-Turetsky)

Let $\mathbf{d}_1, \mathbf{d}_2$ be Turing degrees, with $\mathbf{d}_1$ uniformly Scott complete and $\mathbf{d}_2$ uniformly high for descending sequences. Then $\mathbf{d}_1 \oplus \mathbf{d}_2$ is uniformly high for isomorphism.

Theorem (C.-Franklin-Turetsky)

Let $\mathbf{d}_1, \mathbf{d}_2$ be Turing degrees, with $\mathbf{d}_1$ uniformly Scott complete and $\mathbf{d}_2$ uniformly high for descending sequences. Then $\mathbf{d}_1 \oplus \mathbf{d}_2$ is uniformly high for isomorphism.
Since every degree uniformly high for isomorphism is both uniformly Scott complete and uniformly high for descending sequences, the converse is true, as well.

Proof.

Join the two structures, suspending each structure from a point.

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Compute a Scott analysis.
If the suspension points are equivalent at all standard levels, they must be equivalent at a nonstandard level.
Find a descending sequence in that initial segment, and make a back-and-forth argument along that. □

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Since every degree uniformly high for isomorphism is both uniformly Scott complete and uniformly high for descending sequences, the converse is true, as well.

Theorem

The following properties of a degree **d** *are equivalent:*

1. **d** *computes a completion of Peano arithmetic*
2. *For any infinite computable subtree* T *of* $2^{<\omega}$, *the degree* **d** *computes an infinite path through* T
3. **d** *computes a separating set for any two disjoint c.e. sets.*
4. **d** *is* DNR_2.

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4. **d** *is* DNR_2.

Definition

Such a degree is called a PA degree.

Theorem (C.-Franklin-Turetsky)

Let $\mathbf{d}$ *be PA over* $\mathbf{e}$*, where* $\mathbf{e}$ *is uniformly high for descending sequences. Then* $\mathbf{d}$ *is high for isomorphism.*

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