Degrees High for Isomorphisms

Wesley Calvert

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NERDS 17
Theorem

Let $\mathcal{V}$ be a computable vector space over $\mathbb{Q}$. Then

1. If $\dim(\mathcal{V}) < \aleph_0$, then for every computable $\mathcal{U} \cong \mathcal{V}$ there is a computable isomorphism $f : \mathcal{U} \to \mathcal{V}$. 
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1. If $\dim(\mathcal{V}) < \aleph_0$, then for every computable $\mathcal{U} \cong \mathcal{V}$ there is a computable isomorphism $f : \mathcal{U} \to \mathcal{V}$.

2. If $\dim(\mathcal{V}) = \aleph_0$, then there is a computable $\mathcal{U} \cong \mathcal{V}$ such that no computable function $f : \mathcal{U} \to \mathcal{V}$ is an isomorphism.
Definition

Let $\mathcal{A}$ be a computable structure, and $X \subseteq \mathbb{N}$. 

The $X$-dimension of $\mathcal{A}$ is the number of distinct isomorphic copies $\mathcal{B} \cong \mathcal{A}$ up to $X$-computable isomorphism.
Definition

Let $\mathcal{A}$ be a computable structure, and $X \subseteq \mathbb{N}$.

1. We say that $\mathcal{A}$ is $X$-categorical iff for any computable $\mathcal{B} \cong \mathcal{A}$ there is an $X$-computable isomorphism $f : \mathcal{A} \to \mathcal{B}$. 

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Let $\mathcal{V}$ be a computable vector space over $\mathbb{Q}$. Then

1. $\mathcal{V}$ is $\emptyset'$-categorical

2. $\mathcal{V}$ is $\emptyset$-categorical if and only if $\dim_{\mathbb{Q}} \mathcal{V} < \aleph_0$. 
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Let $\mathcal{V}$ be a computable vector space over $\mathbb{Q}$. Then

1. $\mathcal{V}$ is $\emptyset'$-categorical

2. $\mathcal{V}$ is $\emptyset$-categorical if and only if $\dim_{\mathbb{Q}} \mathcal{V} < \aleph_0$.

3. If $\dim_{\mathbb{Q}} \mathcal{V} = \aleph_0$, then $\dim_{\emptyset} \mathcal{V} = \aleph_0$. 
Let $\mathcal{K}$ be a class of computable structures, and $X \subseteq \mathbb{N}$. 
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**Question**

What are the possible $X$-dimensions of elements of $\mathcal{K}$?
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**Question**

Which elements of $\mathcal{K}$ are computably categorical?

**Question**

Which elements of $\mathcal{K}$ are $X$-categorical?

**Question**

For what $X$ is it true that for every $A \in \mathcal{K}$, the structure $A$ is $X$-categorical?
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**Question**
For what $X$ is it true that for every $A \in \mathcal{K}$, the structure $A$ is $X$-categorical?
Definition

Let $\mathcal{M}$ be a computable structure. The **categoricity spectrum** of $\mathcal{M}$ is the set

$$\{d : \dim_d \mathcal{M} = 1\}.$$
Definition

We say that $\mathcal{A}$ is relatively $X$-categorical if for any $\mathcal{B} \cong \mathcal{A}$, there is an isomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ computable in $\mathcal{B} \oplus X$. 

Problem (Goncharov, et al.)

Let $\mathcal{A}$ be $\Delta^1_1$-categorical. Must $\mathcal{A}$ be relatively $\Delta^1_1$-categorical?
Definition

We say that $\mathcal{A}$ is relatively $X$-categorical if for any $B \cong \mathcal{A}$, there is an isomorphism $f : \mathcal{A} \rightarrow B$ computable in $B \oplus X$.

Problem (Goncharov, et al.)

Let $\mathcal{A}$ be $\Delta_1^1$-categorical. Must $\mathcal{A}$ be relatively $\Delta_1^1$-categorical?
Definition

A degree \( d \) is said to be low for isomorphism if and only if for every pair of computable structures \( A, B \),

- there is \( f : A \to B \) with \( f \leq_T d \) if and only if there is some computable \( g : A \to B \).
Theorem (Franklin–Solomon 2014)

There are degrees $d \not\equiv_T \emptyset$ which are low for isomorphism.
Theorem (Franklin–Solomon 2014)

*There are degrees* $d \not\equiv_T \emptyset$ *which are low for isomorphism.*

Theorem (Franklin–Solomon 2014)

*Every example of the following is low for isomorphism:*

1. Cohen 2-generic
2. Matthias 2-generic
Theorem (Franklin–Solomon 2014)

There exist degrees low for isomorphism with the following combinations of properties:

1. Hyperimmune free and minimal
2. Hyperimmune free and not computably traceable
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Degrees with the following properties are not low for isomorphism:

1. Non-computable $\Delta^0_2$ degrees
2. Degrees computing separating sets for computably inseparable c.e. sets.
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Degrees with the following properties are not low for isomorphism:

1. Non-computable $\Delta^0_2$ degrees
2. Degrees computing separating sets for computably inseparable c.e. sets.

Theorem (Franklin–Solomon 2014)

The set of degrees which are low for isomorphism has measure 0.
Definition

We say that a degree $d$ is the degree of categoricity of $M$ if and only if it is the least degree in the categoricity spectrum of $M$. 
Theorem (Fokina–Kalimullin–Miller 2010, Csima–Franklin–Shore 2012)

If $\alpha$ is any computable ordinal and $\emptyset^{(\alpha)} \leq_T d$ and $d$ is d.c.e. in $\emptyset^{(\alpha)}$, then there is a computable structure $M$ such that $d$ is the degree of categoricity of $M$.
Theorem (Fokina–Kalimullin–Miller 2010, Csima–Franklin–Shore 2012)

If $\alpha$ is any computable ordinal and $\emptyset^{(\alpha)} \leq_T d$ and $d$ is d.c.e. in $\emptyset^{(\alpha)}$, then there is a computable structure $\mathcal{M}$ such that $d$ is the degree of categoricity of $\mathcal{M}$

Problem

Can this be extended to 3-c.e.?
Proposition

*If* $d \geq_T \emptyset$ *is a degree of categoricity, then it is not low for isomorphism.*
Definition
Let $d$ be a degree.
Definition

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1. $d$ is said to be **low** if and only if $d' \equiv_T \emptyset'$.
**Definition**

Let \( d \) be a degree.

1. \( d \) is said to be **low** if and only if \( d' \equiv_T \emptyset' \).
2. \( d \) is said to be **high** if and only if \( d' \equiv_T (\emptyset')' \).
Definition

We say that a degree $d$ is **high for isomorphism** if for any computable structures $\mathcal{A} \cong \mathcal{B}$, there is $f : \mathcal{A} \cong \mathcal{B}$ with $f \leq_T d$. 
**Definition**

We say that a degree $d$ is **high for isomorphism** if for any computable structures $A \cong B$, there is $f : A \cong B$ with $f \leq_T d$.

**Question**

Could such a thing even exist?
Theorem (Scott)

Let $\mathcal{M}$ be a countable structure. Then there is a sentence $\varphi_{\mathcal{M}}$ of $L_{\omega_1 \omega}$ such that for any countable structure $\mathcal{N}$, if $\mathcal{N} \models \varphi_{\mathcal{M}}$, then $\mathcal{N} \cong \mathcal{M}$. 
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Question

Is there a computable structure $\mathcal{M}$ an a sentence $\psi_{\mathcal{M}} \in L_{\omega_1\omega}$ such that
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Is there a computable structure $\mathcal{M}$ an a sentence $\psi_{\mathcal{M}} \in L_{\omega_1 \omega}$ such that

1. For every computable structure $\mathcal{N}$, if $\mathcal{N} \models \psi_{\mathcal{M}}$, then $\mathcal{N} \cong \mathcal{M}$, but
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Is there a computable structure $\mathcal{M}$ an a sentence $\psi_{\mathcal{M}} \in L_{\omega_1 \omega}$ such that

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2. There is a countable structure $\mathcal{N} \not\cong \mathcal{M}$ with $\mathcal{N} \models \psi_{\mathcal{M}}$?

Theorem (Harrison-Trainor 2018)

There is.
Definition

We define the back-and-forth relations on pairs of the form $(\mathcal{A}, \bar{a})$, where $\mathcal{A}$ is a structure and $\bar{a}$ is a finite tuple of elements of $\mathcal{A}$:
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We define the back-and-forth relations on pairs of the form \((\mathcal{A}, \bar{a})\), where \(\mathcal{A}\) is a structure and \(\bar{a}\) is a finite tuple of elements of \(\mathcal{A}\):

1. We say that \((\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})\) if and only if the quantifier-free formulas true of \(\bar{a}\) are also true of \(\bar{b}\).
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2. For \(\alpha \geq 1\), we define \((\mathcal{A}, \vec{a}) \leq_\alpha (\mathcal{B}, \vec{b})\) if and only if for each \(\vec{d}\) in \(\mathcal{B}\) and each \(\beta < \alpha\), there is \(\vec{c} \subseteq \mathcal{A}\) such that \((\mathcal{B}, \vec{b}\vec{d}) \leq_\beta (\mathcal{A}, \vec{a}\vec{c})\).
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We also define \((\mathcal{A}, \bar{a}) \equiv_\alpha (\mathcal{B}, \bar{b})\) if and only if both \((\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})\) and \((\mathcal{B}, \bar{b}) \leq_\alpha (\mathcal{A}, \bar{a})\).
Example

Let $\mathcal{H}_1, \mathcal{H}_2$ be linear orderings of type $\omega_1^{CK}(1 + \mathbb{Q})$. 
Theorem (Nadel 1974)

For any computable structures $\mathcal{A}, \mathcal{B}$, if for all $\alpha < \omega_1^{CK} + 1$ we have $\mathcal{A} \equiv_{\alpha} \mathcal{B}$, then $\mathcal{A} \cong \mathcal{B}$. 
Theorem (Nadel 1974)

For any computable structures $A, B$, if for all $\alpha < \omega_1^{CK} + 1$ we have $A \equiv_\alpha B$, then $A \cong B$.

Corollary

$O'$ is high for isomorphism.
Definition

A $\Sigma^1_1$ class is a set of functions with a definition of the form
\[ \{ f : \mathbb{N} \to \mathbb{N} \mid \exists g \, R(f, g) \} \], where $R$ has only element quantifiers.
Definition

A $\Sigma^1_1$ class is a set of functions with a definition of the form
$$\{ f : \mathbb{N} \rightarrow \mathbb{N} | \exists g \ R(f, g) \}$$
where $R$ has only element quantifiers.

Example

Let $\mathcal{M}_1, \mathcal{M}_2$ be computable structures. Then the class of isomorphisms $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a $\Sigma^1_1$ class.
Theorem (Gandy)

Let $A$ be a $\Sigma^1_1$ class. Then there is $f \in A$ such that $\omega_1^f = \omega_1^{CK}$. 
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Let $A$ be a $\Sigma^1_1$ class. Then there is $f \in A$ such that $\omega^f_1 = \omega^{CK}_1$.

Corollary

$\mathcal{O}$ is high for isomorphism.
Definition

We say that a degree $d$ is uniformly high for isomorphism if and only if there is a single algorithm that will, given a pair of isomorphic computable structures, compute from $d$ an isomorphism.
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Proposition
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Proposition

*If* $d$ *is high for isomorphism, then* $d''$ *is uniformly high for isomorphism.*
Proposition (C.-Franklin-Turetsky)

*Let* $d$ *be a degree.*
Proposition (C.-Franklin-Turetsky)

Let $d$ be a degree.

1. If $d$ is high for isomorphism, then $d''$ enumerates $\mathcal{O}$. 
Proposition (C.-Franklin-Turetsky)

Let $d$ be a degree.

1. If $d$ is high for isomorphism, then $d''$ enumerates $\mathcal{O}$.
2. If $d$ is uniformly high for isomorphism, then $d'$ enumerates $\mathcal{O}$. 
Theorem (C.-Franklin-Turetsky)

There is a degree $d$ which is high for isomorphism with $d''' \equiv_T \emptyset$. 
Proof.

A pointed perfect tree is an increasing function $f : 2^{<\omega} \to 2^{<\omega}$ such that for every path $X$, we have $f \leq_T f(X)$. 
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We build an $O$ computable refining sequence of uniformly pointed perfect trees.
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In even steps, we work to force that $""$ is $O$. 
Proof.

A **pointed perfect tree** is an increasing function $f : 2^{\omega} \rightarrow 2^{\omega}$ such that for every path $X$, we have $f \leq_T f(X)$.

We build an $\mathcal{O}$ computable refining sequence of uniformly pointed perfect trees.

In even steps, we work to force that """ is $\mathcal{O}$.

In odd steps, we work to arrange that this sequence of functions approaches a high for isomorphism.
Corollary

There is a degree $d$ which is uniformly high for isomorphism with $d' \equiv_T \emptyset$. 
Definition
A Harrison ordering is a computable linear ordering isomorphic to $\omega_1^{CK}(1 + \mathbb{Q})$. 
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Definition
We say that a degree $d$ is uniformly Scott complete if $d$ will compute, uniformly in a Harrison ordering $\mathcal{H}$ and a computable structure $\mathcal{A}$, a set of relations $\leq_\alpha$ with the following properties:
Definition

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Definition

We say that a degree $d$ is uniformly Scott complete if $d$ will compute, uniformly in a Harrison ordering $\mathcal{H}$ and a computable structure $\mathcal{A}$, a set of relations $\preceq_\alpha$ with the following properties:

1. If 0 is the least element of $\mathcal{H}$, we say that $(\mathcal{A}, \bar{a}) \preceq_0 (\mathcal{B}, \bar{b})$ if and only if the quantifier-free formulas true of $\bar{a}$ are also true of $\bar{b}$. 
**Definition**

A Harrison ordering is a computable linear ordering isomorphic to \( \omega_1^{CK}(1 + \mathbb{Q}) \).

**Definition**

We say that a degree \( d \) is uniformly Scott complete if \( d \) will compute, uniformly in a Harrison ordering \( \mathcal{H} \) and a computable structure \( A \), a set of relations \( \preceq_\alpha \) with the following properties:

1. If 0 is the least element of \( \mathcal{H} \), we say that \((A, \bar{a}) \preceq_0 (B, \bar{b})\) if and only if the quantifier-free formulas true of \( \bar{a} \) are also true of \( \bar{b} \).

2. For \( \alpha \in \mathcal{H} \) with \( \alpha > 0 \), we define \((A, \bar{a}) \preceq_\alpha (B, \bar{b})\) if and only if for each \( \bar{d} \) in \( B \) and each \( \beta < \alpha \), there is \( \bar{c} \subseteq A \) such that \((B, \bar{b}\bar{d}) \preceq_\beta (A, \bar{a}\bar{c})\).
Definition

Given a Harrison ordering $\mathcal{H}$, a jump structure on $\mathcal{H}$ is a system
\[ \{(x, S_x) : x \in \mathcal{H}\} \]
such that for all $x$, we have $S_x \subseteq \mathbb{N}$ and if $y <_\mathcal{H} x$, we have $S'_y \leq_T S_x$. 
Definition

Given a Harrison ordering $\mathcal{H}$, a jump structure on $\mathcal{H}$ is a system
\[ \{(x, S_x) : x \in \mathcal{H}\} \] such that for all $x$, we have $S_x \subseteq \mathbb{N}$ and if $y <_\mathcal{H} x$, we have $S'_y \leq_T S_x$.

Definition

We say that a degree $d$ is uniformly jump complete if and only if it will compute, uniformly in a Harrison ordering $\mathcal{H}$, a jump structure on $\mathcal{H}$.
Proposition (C.-Franklin-Turetsky)

\[ UHFI \rightarrow UJC \rightarrow USC \]
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Part of Proof.

Suppose \( d \) is UJC.
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\[ UHFI \rightarrow UJC \rightarrow USC \]

Part of Proof.

Suppose \( d \) is UJC.

A single point in the non-well-founded part of a Harrison ordering computes all \( \Delta_\alpha \) relations.
Proposition (C.-Franklin-Turetsky)

There is a uniformly Scott complete degree which is not uniformly jump complete.
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There is a uniformly Scott complete degree which is not uniformly jump complete.

Proof.

There is a uniformly Scott complete degree which is low for $\omega_1^{CK}$. 
Proposition (C.-Franklin-Turetsky)

There is a uniformly Scott complete degree which is not uniformly jump complete.

Proof.

There is a uniformly Scott complete degree which is low for $\omega_1^{CK}$. If $d$ is uniformly jump complete, then $d'' \geq \mathcal{O}$. \qed
Proposition (C.-Franklin-Turetsky)

*Every degree which is uniformly high for isomorphism is uniformly jump complete.*
Proposition (C.-Franklin-Turetsky)

Every degree which is uniformly high for isomorphism is uniformly jump complete.

Proposition (C.-Franklin-Turetsky)

There is a uniformly jump complete degree $d$ with $d'' \equiv_T \emptyset$. 
**Definition**

We say that a degree $d$ is uniformly high for descending sequences if and only if it will compute, uniformly in a Harrison ordering $\mathcal{H}$, a descending sequence in $\mathcal{H}$. 
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Proposition (C.-Franklin-Turetsky)

*Every degree which is uniformly high for isomorphism is uniformly high for descending sequences.*
Theorem (C.-Franklin-Turetsky)

Let $d_1, d_2$ be Turing degrees, with $d_1$ uniformly Scott complete and $d_2$ uniformly high for descending sequences. Then $d_1 \oplus d_2$ is uniformly high for isomorphism.
Theorem (C.-Franklin-Turetsky)

Let $d_1, d_2$ be Turing degrees, with $d_1$ uniformly Scott complete and $d_2$ uniformly high for descending sequences. Then $d_1 \oplus d_2$ is uniformly high for isomorphism. 

Since every degree uniformly high for isomorphism is both uniformly Scott complete and uniformly high for descending sequences, the converse is true, as well.
Proof.
Join the two structures, suspending each structure from a point.
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Join the two structures, suspending each structure from a point. Compute a Scott analysis.
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Join the two structures, suspending each structure from a point. Compute a Scott analysis.
If the suspension points are equivalent at all standard levels, they must be equivalent at a nonstandard level.
Proof.
Join the two structures, suspending each structure from a point. Compute a Scott analysis. If the suspension points are equivalent at all standard levels, they must be equivalent at a nonstandard level. Find a descending sequence in that initial segment, and make a back-and-forth argument along that.
Theorem (C.-Franklin-Turetsky)

Let $d_1, d_2$ be Turing degrees, with $d_1$ uniformly Scott complete and $d_2$ uniformly high for descending sequences. Then $d_1 \oplus d_2$ is uniformly high for isomorphism.
Alternate Characterizations

Theorem (C.-Franklin-Turetsky)

Let $d_1, d_2$ be Turing degrees, with $d_1$ uniformly Scott complete and $d_2$ uniformly high for descending sequences. Then $d_1 \oplus d_2$ is uniformly high for isomorphism.

Since every degree uniformly high for isomorphism is both uniformly Scott complete and uniformly high for descending sequences, the converse is true, as well.
Theorem

The following properties of a degree $d$ are equivalent:

1. $d$ computes a completion of Peano arithmetic
2. For any infinite computable subtree $T$ of $2^{<\omega}$, the degree $d$ computes an infinite path through $T$
3. $d$ computes a separating set for any two disjoint c.e. sets.
4. $d$ is $DNR_2$. 
Theorem

The following properties of a degree $d$ are equivalent:

1. $d$ computes a completion of Peano arithmetic
2. For any infinite computable subtree $T$ of $2^{<\omega}$, the degree $d$ computes an infinite path through $T$
3. $d$ computes a separating set for any two disjoint c.e. sets.
4. $d$ is DNR$_2$.

Definition

Such a degree is called a PA degree.
Theorem (C.-Franklin-Turetsky)

Let $d$ be PA over $e$, where $e$ is uniformly high for descending sequences. Then $d$ is high for isomorphism.
Degrees High for Isomorphisms

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