Analytic complete equivalence relations and their degree spectra

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There are several notions of computational complexity for a structure. One of the most important ones are spectra.

1. Given $\mathcal{A}$, the **atomic diagram** $D(\mathcal{A})$ is the set of basic formulas true of $\mathcal{A}$. Then the **isomorphism spectrum** of $\mathcal{A}$ is

$$DgSp_{\cong}(\mathcal{A}) = \{deg(D(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\}.$$ 

2. Let $\mathcal{A}$ be computable (i.e., $D(\mathcal{A})$ is computable), then the **categoricity spectrum** of $\mathcal{A}$ is

$$CatSp_{\cong}(\mathcal{A}) = \{deg(f) : f : \mathcal{A} \cong \mathcal{B}, \mathcal{B} \text{ computable}\}.$$ 

These notions have received much attention over last decades.

Typical question: What families of degrees are realized as spectra?
Definition (Fokina, Semukhin, Turetsky; Montalbán; Yu)
Let $E$ be an invariant equivalence relation on a class of structures $\mathcal{C}$. The $E$-degree spectrum of $\mathcal{A} \in \mathcal{C}$ is

$$DgSp_E(\mathcal{A}) = \{\text{deg}(D(B)) : B \in \mathcal{E}\}.$$

Definition
Let $E$ be an invariant equivalence relation on a class of structures $\mathcal{C}$. The $E$-categoricity spectrum of a computable structure $\mathcal{A}$ is

$$CatSp_E(\mathcal{A}) = \{\text{deg}(f) : f \text{ witnesses } \mathcal{A} \in \mathcal{E}, \mathcal{B} \text{ computable}\}$$

“witnesses” is intentionally ubiquitous, depends on $E$
Examples of equivalence relations

- Elementary equivalence $\mathcal{A} \equiv \mathcal{B}$ (Andrews, J.Miller; Andrews, Knight; ACDLM)

Question:

Let $X$ be a $\equiv$, $\approx$, $\not\approx$-spectrum, is $X' = \{d' : d \in X\}$? (positive for $\sim$)

Question:

Let $a \mid b$, is $\{d \geq a\} \cup \{d \geq b\}$ an $\equiv$ spectrum? (positive for $\equiv$, $\equiv_{\geq 2}$, negative for $\sim =$, $\not\equiv$)
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- $\Sigma_n$ equivalence $\equiv_n$ (structures satisfying the same f.o. $\Sigma_n$ sentences) (Fokina, Semukhin, Turetsky),
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- $\mathcal{A}$ embeds into $\mathcal{B}$, $\mathcal{A} \hookrightarrow \mathcal{B}$, if $\mathcal{A}$ is isomorphic to a substructure of $\mathcal{B}$
  $\mathcal{A}$ is bi-embeddable with $\mathcal{B}$, $\mathcal{A} \approx \mathcal{B}$, if $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$ (Fokina, R., San Mauro ($\approx$-spectra)) (Bazhenov, FRSM ($\approx$-categoricity spectra)),

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  and $\mathcal{A}$ is elementary bi-embeddable with $\mathcal{B}, \mathcal{A} \approx \mathcal{B}$ if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{A}$ (R. ($\approx$-spectra)).
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Question: Let $a | b$, is $\{d \geq a\} \cup \{d \geq b\}$ a $\approx$ spectrum? (positive for $\equiv$, $\equiv_{n \geq 2}$, negative for $\approx$, $\approx$)
## Relationship between degree spectra

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$\cong$, $\approx$, $\equiv$ are not Borel.

$\approx$, $\cong$ are $\Sigma^1_1$-complete.
Spectra preserving reductions

Definition (cf. HTMMM, MPSS)
Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An $(\alpha)$-jump functor between $\mathcal{C}$ and $\mathcal{D}$ is a pair of computable operators $(\Phi, \Phi_*)$ such that

1. for all $A \in \mathcal{C}_1$, $F(A) = \Phi A^{(\alpha)}$,
2. for all $f : A \to B \in \mathcal{C}_2$, $F(f) = \Phi A^{(\alpha)} \oplus f^{(\alpha)} \oplus B^{(\alpha)}$.

If $\mathcal{C}$ and $\mathcal{D}$ are groupoids (all arrows invertible), then we can define the relationship between categories.

Definition
Two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \supseteq \text{Sat}_{\mathcal{D}_2}(F(\mathcal{C})) \to \mathcal{C}$ are (effectively) pseudo inverse if there are (computable) operators $\Lambda_{\mathcal{C}}$ and $\Lambda_{\mathcal{D}}$ such that for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$

$$\Lambda_{\mathcal{C}}^A : A \to G(F(A)) \text{ and } \Lambda_{\mathcal{D}}^B : B \to F(G(B)).$$

We call $G$ an (effective) pseudo inverse of $F$.

We say that $F$ and $G$ are jump effectively pseudo inverses if we require the jump of a structure as oracle in at least one of the $\Lambda$. 
Examples of categories:

- \( \mathcal{G} \cong \) where objects are graphs and arrows isomorphisms,
- \( \mathcal{G} \approx \) where objects are graphs and arrows are pairs of embeddings \((f, g)\),
- \( \mathfrak{A} \equiv \) where objects are abelian groups and arrows are pairs \((\mathfrak{A}, \mathfrak{B})\) indicating that \( \mathfrak{A} \) and \( \mathfrak{B} \) are elementary equivalent.

**Proposition**

Let \( \mathcal{C} \) be given by an equivalence relation \( \sim_1 \) and \( \mathcal{D} \) be given by an equivalence relation \( \sim_2 \), then

1. if there is computable \( F : \mathcal{C} \to \mathcal{D} \) with an \((\alpha)\)-jump pseudo-inverse, then for every \( \mathfrak{A} \in \mathcal{C} \)

   \[
   DgSp_{\sim_2}(F(\mathfrak{A})) = \{ d : d^{(\alpha)} \in DgSp_{\sim_1}(\mathfrak{A}) \},
   \]

2. and if \( F \) has an \((\alpha)\)-jump effective pseudo-inverse, then

   \[
   CatSp_{\sim_2}(F(\mathfrak{A})) = \{ d : d^{(\alpha)} \in CatSp_{\sim_1}(\mathfrak{A}) \}.
   \]
Theorem (R.)

There is a functor $F : \mathcal{G} \approx \rightarrow \mathcal{G} \cong$ with a jump computable jump effective pseudo-inverse.

Proof Idea.
We give a functor $\hat{F} : \mathcal{G} \rightarrow \mathcal{G} \cong$ as the composition of two functors:

\[
\begin{array}{cccc}
\mathcal{G} & \rightarrow & H & \rightarrow & \mathcal{G} \cong \\
& & & & \\
& & & & \text{(R. '18)}
\end{array}
\]

$\hat{F}$ induces the functor $F : \mathcal{G} \approx \rightarrow \mathcal{G} \cong$ with the desired properties.
Given $A, B \in \mathcal{G}$ we want to define $H$ such that

$$A \hookrightarrow B \iff H(A) \preceq H(B).$$

**Idea:** Given $i : A \hookrightarrow B$ but $A \not\preceq B$ there is an $\bar{a} \in A^{<\omega}$ and a formula $\varphi(\bar{x})$ with $A \models \varphi(\bar{a})$ but $B \not\models \varphi(i(\bar{a}))$.

Code the edge relation so that $\varphi$ gets “pushed” out of the f.o. theory.
The functor $H : \mathcal{G} \rightarrow \Gamma \leq$

Given $\mathcal{A}, \mathcal{B} \in \mathcal{G}$ we want to define $H$ such that

$$\mathcal{A} \leftrightarrow \mathcal{B} \iff H(\mathcal{A}) \leq H(\mathcal{B}).$$

Idea: Given $\iota : \mathcal{A} \leftrightarrow \mathcal{B}$ but $\mathcal{A} \not\equiv \mathcal{B}$ there is an $\bar{a} \in A^{\leq \omega}$ and a formula $\varphi(\bar{x})$ with $\mathcal{A} \models \varphi(\bar{a})$ but $\mathcal{B} \not\models \varphi(\iota(\bar{a})).$

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We know how to do that! Marker extensions (Pairs of structures)
The functor $H : \mathcal{G} \rightarrow \Gamma_{\leq}$

Given $\mathcal{A}, \mathcal{B} \in \mathcal{G}$ we want to define $H$ such that

$$\mathcal{A} \leftrightarrow \mathcal{B} \Leftrightarrow H(\mathcal{A}) \preceq H(\mathcal{B}).$$

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We know how to do that! Marker extensions (Pairs of structures)

Replace edges with copies of a structure $\mathcal{C}$ and non-edges $\mathcal{D}$.

For example:

$\mathcal{A} : a \rightarrow b \quad \triangleright \quad \mathcal{g}(\mathcal{A}) : a^g \quad \mathcal{D}$

$\mathcal{C}$

$\triangleright \quad b^g$
At a minimum we need $\mathcal{C} \equiv \mathcal{D}$. 
At a minimum we need $C \equiv D$. To obtain a pseudo-inverse we need $C$ and $D$ with special properties:

**Definition**

1. A structure $\mathcal{A}$ is **weakly minimal**, if $\mathcal{B} \preceq \mathcal{A}$ implies $\mathcal{B} \cong \mathcal{A}$.
2. A structure $\mathcal{A}$ is **minimal**, if there is no $\mathcal{B}$ such that $\mathcal{B} \preceq \mathcal{A}$.

**Question (Vaught):** What is the number of minimal models a theory can have?

**Theorem (Fuhrken, Shelah)**

*For every $\kappa \in \omega \cup \{\aleph_0, 2^{\aleph_0}\}$ there is a theory with $\kappa$ minimal models.*
Shelah’s theory

For $\nu \in 2^{<\omega}$ define $F_\nu : 2^\omega \to 2^\omega$, $\sigma \mapsto \nu +_2 \sigma$ (where $\nu$ is interpreted as $\nu \sim 0$ and $+_2$ is base 2 addition).

Let $R_\nu = \{\sigma \in 2^\omega : \nu \leq \sigma\}$ and consider the theory $T$ of

$$\mathcal{A} = (2^\omega, \langle F_\nu \rangle_{\nu \in 2^{<\omega}}, \langle R_\nu \rangle_{\nu \in 2^{<\omega}}).$$

Shelah used $T$ and variations of $T$ to prove his theorem. It is easy to see that

1. $T$ has quantifier elimination,
2. the substructure $\langle \sigma \rangle$ generated by $\sigma \in 2^\omega$ is an elementary substructure of $\mathcal{A}$,
3. $\langle \sigma \rangle$ is minimal,
4. if $\sigma \nleq \tau$ and $\tau \nleq \sigma$, then there is a $\Sigma^c_2$ sentence distinguishing $\langle \sigma \rangle$ and $\langle \tau \rangle$. 
Using $C = \langle 0 \rangle$ and $D = \langle 1 \rangle$ we obtain a computable functor $G \hookrightarrow \rightarrow \Gamma \approx$ and an induced computable functor $G \approx \rightarrow \Gamma \approx$ with an effective pseudo inverse.

The pseudo inverse is jump computable, but not computable. Thus composing the functors we get a functor $F : G \approx \rightarrow G \approx$ with a jump computable effective pseudo inverse.

**Corollary**

1. *For every graph* $\mathcal{A}$, $DgSp\approx(F(\mathcal{A})) = \{d : d' \in DgSp\approx(\mathcal{A})\}$.

2. *The elementary bi-embeddability relation on graphs is $\Sigma_1^1$-complete.*

Work in progress: Relationship between $CatSp\approx$ and $CatSp\approx$. 
Conclusion and open questions

It appears that we can not improve the result using Marker extensions, i.e., we can not get a computable functor with computable pseudo-inverse with this technique.

This result would require new techniques.

Question

Is every jump of an elementary bi-embeddability spectrum an elementary bi-embeddability spectrum, i.e., if $X$ is an elementary bi-embeddability spectrum, is $X' = \{d' : d \in X\}$?

This would imply that no bi-embeddability spectrum can be the union of two cones.
Thank you!