Analytic complete equivalence relations and their degree spectra

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Complexity of structures

There are several notions of computational complexity for a structure. One of the most important ones are spectra.

1. Given $\mathcal{A}$, the atomic diagram $D(\mathcal{A})$ is the set of basic formulas true of $\mathcal{A}$. Then the isomorphism spectrum of $\mathcal{A}$ is

$$DgSp_{\cong}(\mathcal{A}) = \{deg(D(\mathcal{B})) : \mathcal{B} \cong \mathcal{A}\}.$$

2. Let $\mathcal{A}$ be computable (i.e., $D(\mathcal{A})$ is computable), then the categoricity spectrum of $\mathcal{A}$ is

$$CatSp_{\cong}(\mathcal{A}) = \{deg(f) : f : \mathcal{A} \cong \mathcal{B},\ \mathcal{B} \text{ computable}\}.$$

These notions have received much attention over last decades.

Typical question: What families of degrees are realized as spectra?

Definition (Fokina, Semukhin, Turetsky; Montalbán; Yu)

Let E be an invariant equivalence relation on a class of structures $\mathfrak{C}$. The E-degree spectrum of $\mathcal{A} \in \mathfrak{C}$ is

$$DgSp_E(\mathcal{A}) = \{deg(D(\mathcal{B})) : \mathcal{B}\, E\, \mathcal{A}\}.$$

Definition

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$$CatSp_E(\mathcal{A}) = \{deg(f) : f \text{ witnesses } \mathcal{A}\, E\, \mathcal{B}, \mathcal{B}\text{computable}\}$$

"witnesses" is intentionally ubiquous, depends on E

Examples of equivalence relations

- Elementary equivalence $\mathcal{A} \equiv \mathcal{B}$ (Andrews, J.Miller; Andrews, Knight; ACDLM)

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- $\mathcal{A}$ embeds into $\mathcal{B}$,$\mathcal{A} \hookrightarrow \mathcal{B}$, if $\mathcal{A}$ is isomorphic to a substructure of $\mathcal{B}$

  $\mathcal{A}$ is bi-embeddable with $\mathcal{B}$, $\mathcal{A} \approx \mathcal{B}$, if $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$ (Fokina, R., San Mauro ($\approx$-spectra)) (Bazhenov, FRSM ($\approx$-categoricity spectra)),

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Question: Let X be a $\equiv$, $\approx$, $\approxeq$-spectrum, is $X' = \{\mathbf{d}' : \mathbf{d} \in X\}$? (positive for $\cong$)

Question: Let $\mathbf{a} \mid \mathbf{b}$, is $\{\mathbf{d} \geq \mathbf{a}\} \cup \{\mathbf{d} \geq \mathbf{b}\}$ a $\approx$ spectrum? (positive for $\equiv$, $\equiv_{n\geq 2}$, negative for $\cong$, $\approxeq$)

Relationship between degree spectra

	$\subseteq$	$\cong$	$\approx$	$\approxeq$	$\equiv$	$\equiv_n$
	$\cong$	✓	?	?	✗	✗
(FRS)	$\approx$	?	✓	?	✗	✗
(R)	$\approxeq$	?	?	✓	✗	✗
(AM; AK)	$\equiv$	✗	?	✗	✓	✗
(FST) for fixed n>1	$\equiv_n$	✗	?	✗	✗	✓

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(R)	$\approxeq$	?	?	✓	✗	✗
(AM; AK)	$\equiv$	✗	?	✗	✓	✗
(FST) for fixed n>1	$\equiv_n$	✗	?	✗	✗	✓

$\cong$, $\approx$, $\approxeq$ are not Borel.

$\approx$, $\approxeq$ are $\boldsymbol{\Sigma}^1_1$-complete.

Definition (cf. HTMMM, MPSS)

Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. An (α)-jump functor between $\mathfrak{C}$ and $\mathfrak{D}$ is a pair of computable operators (Φ, Φ_*) such that

1. for all $\mathcal{A} \in \mathfrak{C}_1$, $F(\mathcal{A}) = \Phi^{\mathcal{A}^{(\alpha)}}$,
2. for all $f : \mathcal{A} \to \mathcal{B} \in \mathfrak{C}_2$, $F(f) = \Phi^{\mathcal{A}^{(\alpha)} \oplus f^{(\alpha)} \oplus \mathcal{B}^{(\alpha)}}$.

If $\mathfrak{C}$ and $\mathfrak{D}$ are groupoids (all arrows invertible), then we can define the relationship between categories.

Definition

Two functors $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \supseteq Sat_{\mathfrak{D}_2}(F(\mathfrak{C})) \to \mathfrak{C}$ are (effectively) pseudo inverse if there are (computable) operators $\Lambda_{\mathfrak{C}}$ and $\Lambda_{\mathfrak{D}}$ such that for all $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{B} \in \mathfrak{D}$

$$\Lambda_{\mathfrak{C}}^{\mathcal{A}} : \mathcal{A} \to G(F(\mathcal{A})) \text{ and } \Lambda_{\mathfrak{D}}^{\mathcal{B}} : \mathcal{B} \to F(G(\mathcal{B})).$$

We call G an (effective) pseudo inverse of F.

We say that F and G are jump effectively pseudo inverses if we require the jump of a structure as oracle in at least one of the Λ.

Examples of categories:

- $\mathfrak{G}^{\cong}$ where objects are graphs and arrows isomorphisms,
- $\mathfrak{G}^{\approx}$ where objects are graphs and arrows are pairs of embeddings (f, g),
- $\mathfrak{A}^{\equiv}$ where objects are abelian groups and arrows are pairs $(\mathcal{A}, \mathcal{B})$ indicating that $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent.

Proposition

Let $\mathfrak{C}$ be given by an equivalence relation $\sim_1$ and $\mathfrak{D}$ be given by an equivalence relation $\sim_2$, then

1. *if there is computable $F : \mathfrak{C} \to \mathfrak{D}$ with an (α)-jump pseudo-inverse, then for every $\mathcal{A} \in \mathfrak{C}$*

$$DgSp_{\sim_2}(F(\mathcal{A})) = \{\mathbf{d} : \mathbf{d}^{(\alpha)} \in DgSp_{\sim_1}(\mathcal{A})\},$$

2. *and if F has an (α)-jump effective pseudo-inverse, then*

$$CatSp_{\sim_2}(F(\mathcal{A})) = \{\mathbf{d} : \mathbf{d}^{(\alpha)} \in CatSp_{\sim_1}(\mathcal{A})\}.$$

Theorem (R.)

There is a functor $F : \mathfrak{G}^{\approx} \to \mathfrak{G}^{\approxeq}$ with a jump computable jump effective pseudo-inverse.

Proof Idea.

We give a functor $\hat{F} : \mathfrak{G}^{\hookrightarrow} \to \mathfrak{G}^{\preccurlyeq}$ as the composition of two functors:

$$\mathfrak{G}^{\hookrightarrow} \xrightarrow[\text{H}]{\hspace{4cm}} \Gamma^{\preccurlyeq} \xrightarrow[\text{R. '18}]{\hspace{4cm}} \mathfrak{G}^{\preccurlyeq}$$

$\hat{F}$ induces the functor $F : \mathfrak{G}^{\approx} \to \mathfrak{G}^{\approxeq}$ with the desired properties.

Given $\mathcal{A}, \mathcal{B} \in \mathfrak{G}$ we want to define H such that

$$\mathcal{A} \hookrightarrow \mathcal{B} \Leftrightarrow H(\mathcal{A}) \preccurlyeq H(\mathcal{B}).$$

Idea: Given $\iota : \mathcal{A} \hookrightarrow \mathcal{B}$ but $\mathcal{A} \not\preccurlyeq \mathcal{B}$ there is an $\bar{a} \in A^{<\omega}$ and a formula $\varphi(\bar{x})$ with $\mathcal{A} \models \varphi(\bar{a})$ but $\mathcal{B} \not\models \varphi(\iota(\overline{a}))$.

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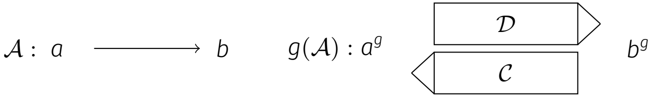
Replace edges with copies of a structure $\mathcal{C}$ and non-edges $\mathcal{D}$.

For example:



At a minimum we need $\mathcal{C} \equiv \mathcal{D}$.

At a minimum we need $\mathcal{C} \equiv \mathcal{D}$. To obtain a pseudo-inverse we need $\mathcal{C}$ and $\mathcal{D}$ with special properties:

Definition

1. A structure $\mathcal{A}$ is weakly minimal, if $\mathcal{B} \preccurlyeq \mathcal{A}$ implies $\mathcal{B} \cong \mathcal{A}$.
2. A structure $\mathcal{A}$ is minimal, if there is no $\mathcal{B}$ such that $\mathcal{B} \precneqq \mathcal{A}$.

Question (Vaught): What is the number of minimal models a theory can have?

Theorem (Fuhrken, Shelah)

For every $\kappa \in \omega \cup \{\aleph_0, 2^{\aleph_0}\}$ there is a theory with κ minimal models.

For $\nu \in 2^{<\omega}$ define $F_\nu : 2^\omega \to 2^\omega$, $\sigma \mapsto \nu +_2 \sigma$ (where ν is interpreted as $\nu^\frown \bar{0}$ and $+_2$ is base 2 addition).

Let $R_\nu = \{\sigma \in 2^\omega : \nu \preceq \sigma\}$ and consider the theory T of

$$\mathcal{A} = (2^\omega, \langle F_\nu \rangle_{\nu \in 2^{<\omega}}, \langle R_\nu \rangle_{\nu \in 2^{<\omega}}).$$

Shelah used T and variations of T to prove his theorem. It is easy to see that

1. T has quantifier elimination,
2. the substructure $\langle \sigma \rangle$ generated by $\sigma \in 2^\omega$ is an elementary substructure of $\mathcal{A}$,
3. $\langle \sigma \rangle$ is minimal,
4. if $\sigma \not\preceq \tau$ and $\tau \not\preceq \sigma$, then there is a Σ_2^c sentence distuingishing $\langle \sigma \rangle$ and $\langle \tau \rangle$.

Using $\mathcal{C} = \langle \bar{0} \rangle$ and $\mathcal{D} = \langle \bar{1} \rangle$ we obtain a computable functor $\mathfrak{G}^{\hookrightarrow} \to \Gamma^{\preccurlyeq}$ and and induced computable functor $\mathfrak{G}^{\approx} \to \Gamma^{\approxeq}$ with an effective pseudo inverse.

The pseudo inverse is jump computable, but not computable. Thus composing the functors we get a functor $F : \mathfrak{G}^{\approx} \to \mathfrak{G}^{\approxeq}$ with a jump computable effective pseudo inverse.

Corollary

1. *For every graph $\mathcal{A}$, $DgSp_{\approxeq}(F(\mathcal{A})) = \{\mathbf{d} : \mathbf{d}' \in DgSp_{\approx}(\mathcal{A})\}$.*
2. *The elementary bi-embeddability relation on graphs is $\mathbf{\Sigma}^1_1$-complete.*

Work in progress: Relationship between $CatSp_{\approx}$ and $CatSp_{\approxeq}$.

It appears that we can not improve the result using Marker extensions, i.e., we can not get a computable functor with computable pseudo-inverse with this technique.

This result would require new techniques.

Question

Is every jump of an elementary bi-embeddability spectrum an elementary bi-embeddability spectrum, i.e., if X is an elementary bi-embeddability spectrum, is $X' = \{\mathbf{d}' : \mathbf{d} \in X\}$?

This would imply that no bi-embeddability spectrum can be the union of two cones.

Thank you!