Borel combinatorics goes wrong in HYP

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Joint work with Rose Weisshaar and Linda Brown Westrick

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One example is the Borel sets, which require some coding to make sense of: to even proof some basic facts about codes for Borel sets, we already need the theory ATR_0.

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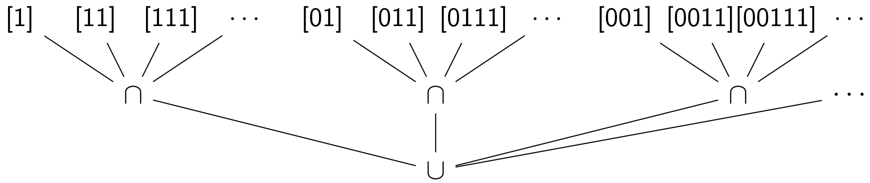
A Borel set is a subset of $\{0,1\}^\omega$ where:

- each clopen set $[\sigma] = \{X \mid \forall i < |\sigma|\ X(i) = \sigma(i)\}$ is Borel, and
- a countable union or intersection or Borel sets is Borel.

A *Borel code* is a *tree*—that is, a set of sequences closed under subsequences—such that:

- the tree is well-founded,
- each leaf encodes a basic clopen subset of $\{0,1\}^\omega$,
- each interior node encodes either a union or an intersection.

For instance:



encodes the set of sequences of the form $0000\cdots 01111\cdots$.

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We would normally do this by transfinite induction: we should be able to tell which (sets encoded by) leaves X belongs to and then we can tell if X belongs to (the set encoded by) a node by asking whether it belongs to some/all of the children.

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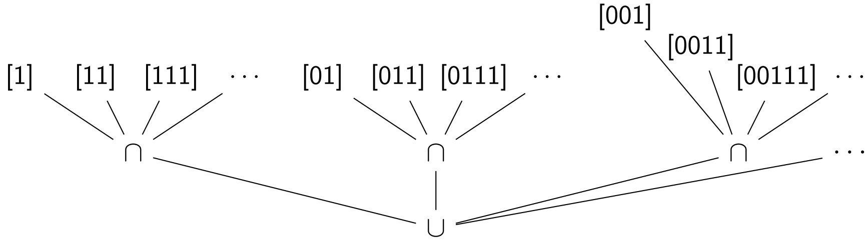

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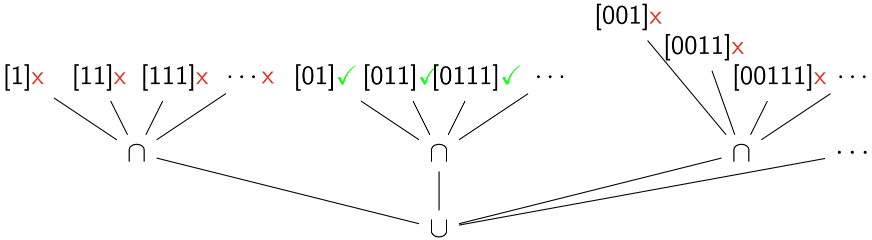

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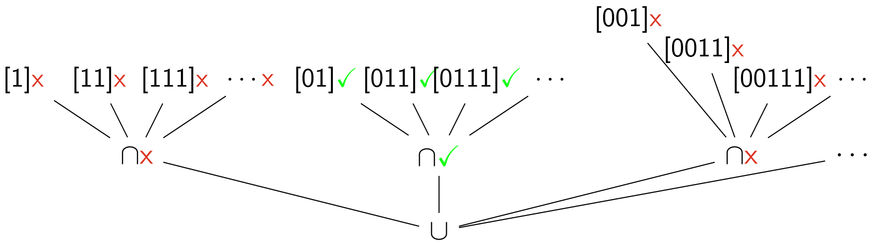

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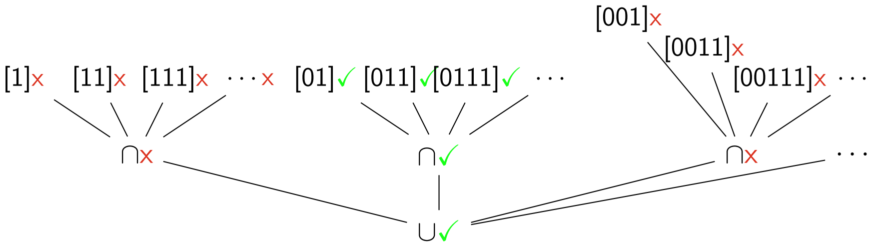

we can:

1. check which leaves it belongs to,
2. check which nodes at the next level it belongs to,
3. check if it belongs to the root.

Definition

When T is a Borel code, an *evaluation map* for X is a function $f : T \to \{0, 1\}$ such that:

- if σ is a leaf, $f(\sigma) = 1$ if and only if X is in the clopen set encoded at σ,
- if σ is a union node, $f(\sigma) = 1$ if and only if $f(\sigma^\frown n)$ for some n,
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An evaluation map must be unique, so we can say:

- $X \in T$ if there is an evaluation map f so that $f(\langle\rangle) = 1$,
- $X \notin T$ if there is an evaluation map f so that $f(\langle\rangle) = 0$.

But an evaluation map is itself (encoded by) a set, and we need to be able to prove they exist. Constructing an evaluation map requires a transfinite recursion—which is exactly what we need ATR_0 for:

Theorem

Working over RCA_0, if, for every Borel code T and every X, an evaluation map for X exists, then ATR_0 holds.

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Even "every Borel code is either non-empty, or has a non-empty complement" already implies ATR_0.

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When trying to measure the strength of various theorems about Borel sets, maybe the correct hypothesis to work with is not "for every Borel set" but instead "for every completely determined Borel set".

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This puts these principles in a range that includes the “theories of hyperarithmetic analysis”—theories closely linked to what happens in ω-models whose sets are exactly $HYP(Y)$ for some Y.

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Do statements about completely determined Borel sets behave reasonably in HYP?

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But HYP contains *pseudo-wellorders*: there are orderings α^* which are ill-founded, but where there is no hyperarithmetic decreasing sequence. So

$$HYP \vDash \text{ there are Borel codes which are not completely determined}$$

because there are trees T^* which HYP "thinks" are Borel codes, but which are actually ill-founded and fail to have hyperarithmetic evaluation maps.

ADMSW introduced the "decorating trees" method for constructing defective Borel codes in HYP—that is, trees which:

- are hyperarithmetic,
- have pseudo-well-founded height (i.e. are ill-founded, but appear to be well-founded in HYP),
- are completely determined in HYP.

As we'll see, this method will let us construct a variety of very strange "Borel codes" in HYP.

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For any particular pair $(X, Y) \in HYP$, there is a least β such that $X, Y \leq \emptyset^\beta$. The set of such pairs is a set S_β, and there is a Borel code for this set of height roughly β.

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Furthermore, within S_β, the set of pairs (X, Y) with $X < Y$ is a set R_β which is also encoded by a Borel code of height roughly β.

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The codes for S_β, R_β are given uniformly in β, so we can make sense of S_{β^*}, R_{β^*} for pseudo-well-ordinals as well.

For formal reasons, we assume all our Borel codes alternate union and intersection levels and are associated with a rank function ρ which assigns a pseudo-ordinal to each node.

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- `Decorate`(T) of a leaf leaves the leaf unchanged,

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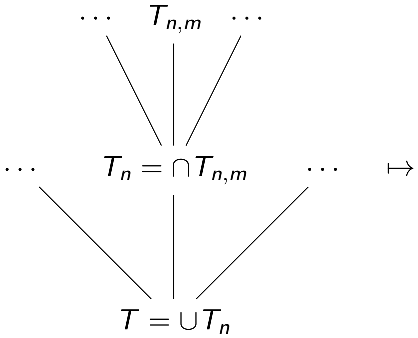
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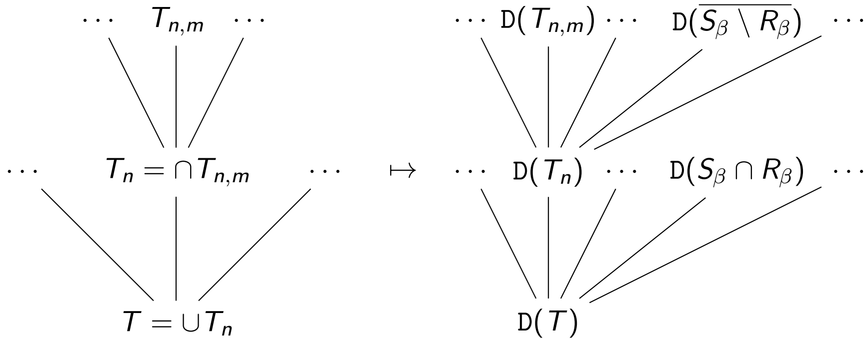
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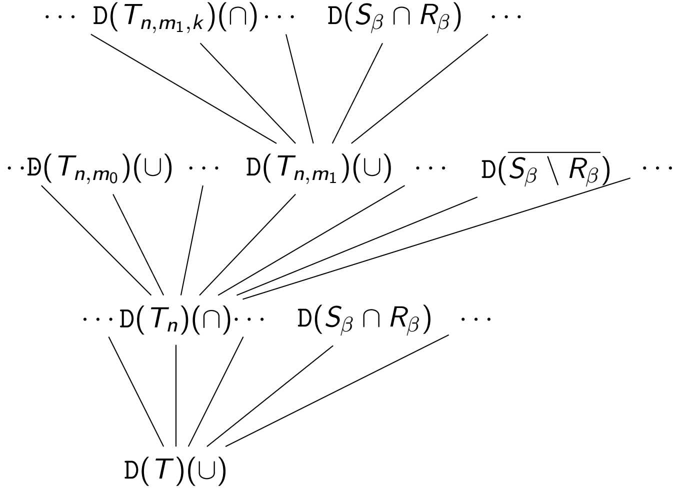
That is, we can mark all the high rank nodes uniformly in a bit more than β jumps.

Suppose $X < Y$, $(X, Y) \in S_\beta$ and T, T_{n,m_1}, $T_{n,m_1,k}$ are much taller than β while T_{n,m_0} has height less than β. Then:



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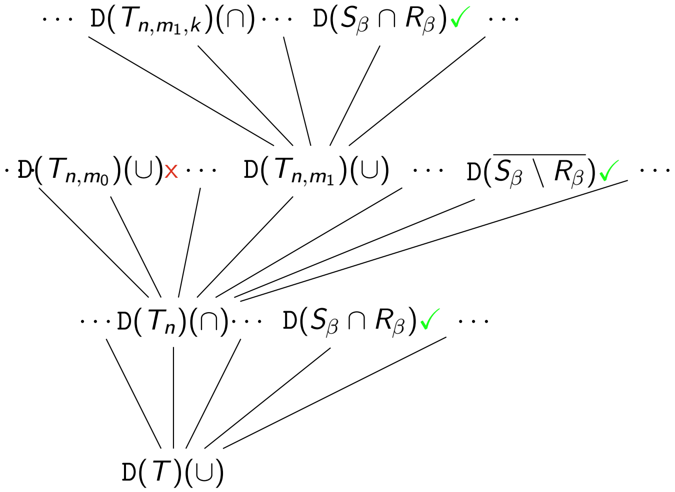

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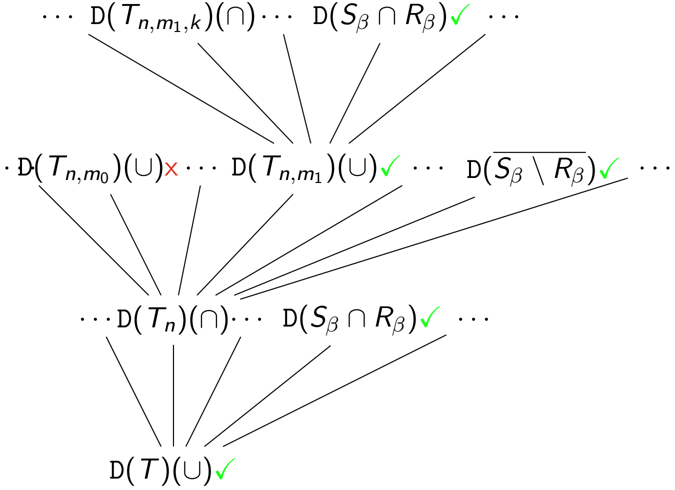

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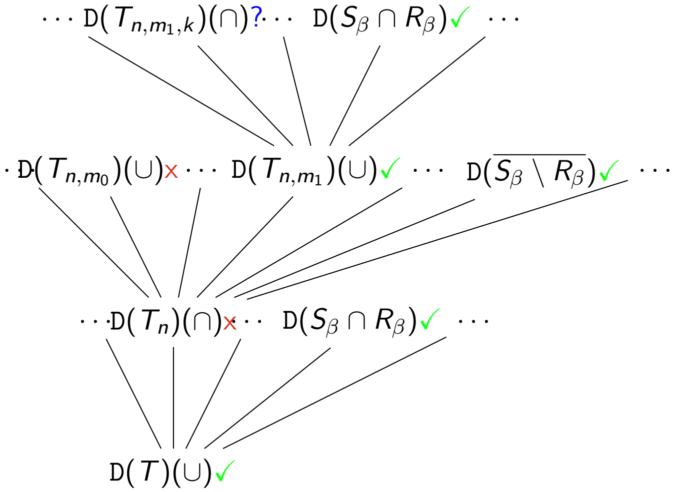

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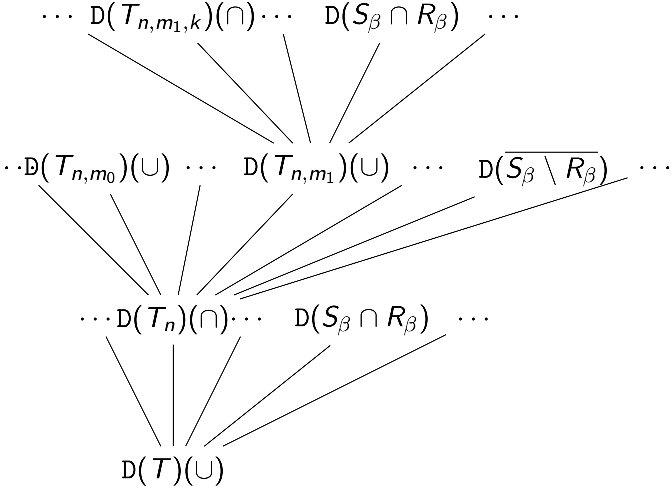

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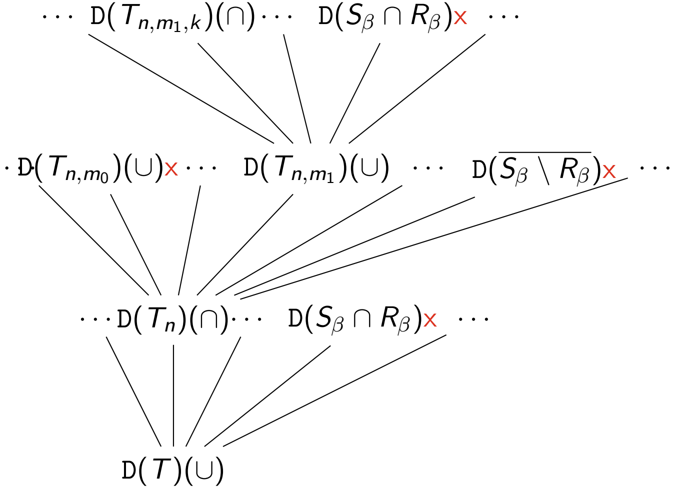

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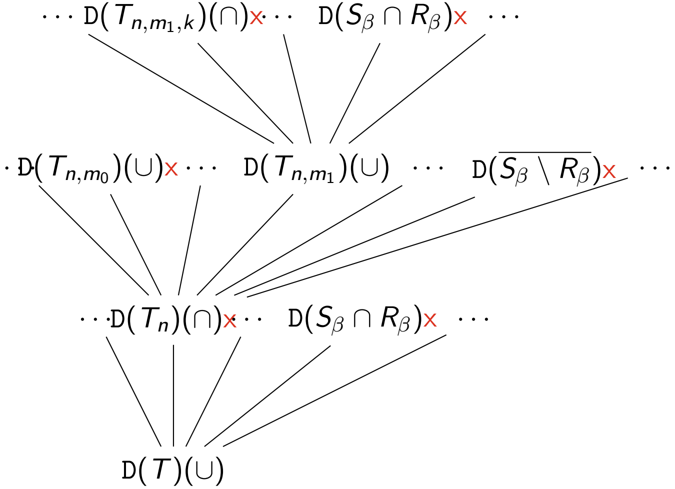

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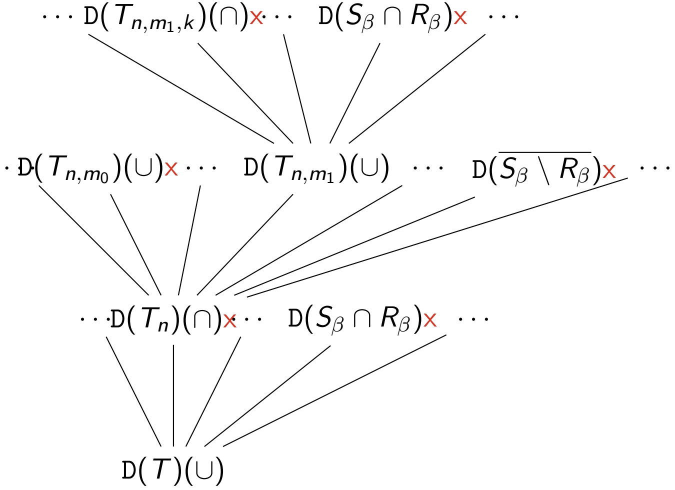

Putting this together with checking that this "Borel code" really does compute the well-ordering, we conclude that, in HYP:

- this is a Borel code T,
- it is completely determined, and
- for any (X, Y), there is a unique evaluation map $ev_T(X, Y)$ which is 1 if and only if $X < Y$.

The main idea is that we can, in a uniform and hyperarithmetic way, stratify the pairs of hyperarithmetic reals into the levels S_β.

We then define our tree so that whatever happens, it happens very uniformly: we can create an evaluation map for the low rank part of the tree in a hyperarithmetic way, and because we've made copies of it everywhere, we know we can define the evaluation map blindly on the rest of it.

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This theorem is also true in HYP...but not for the right reason.

For instance, the graph whose vertices are increasing sequences of natural numbers and there is an edge between X and Y exactly when, for all n, $X(n) = Y(n+1)$ (or vice-versa).

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- Our stratification is to take S_β to be those X so that β is least with $X \leq \emptyset^\beta$.
- If e is such that, for all $e' < e$, $(e')^\beta$ is in a different component from e, then we make 0 the color of $(e)^\beta$; we color the rest of the component based on distance from this point. This can be described by some Borel code R_β of height roughly β.

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HYP satisfies "if G is a completely determined, Borel, d-regular acyclic graph for some finite d then G has a completely determined Borel 2-coloring".

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The idea is basically the same. Because we know each real has exactly d neighbors, each real has a hyperarithmetic connected component (complete with evaluation maps). We can color each X by looking for the first (in the sense of the well-ordering above) Y encoding the connected component of X, coloring the first column of Y 0, and coloring X based on its distance to that point.

For those X whose first encoding of the connected component is $\emptyset^{\beta}$-computable, we can do this with an actual Borel code R_{β}.

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If β is least such that there are $\emptyset^\beta$-computable evaluation maps for $(e)^{\emptyset^\alpha}$, we choose either one or two $\emptyset^\beta$-computable reals and make a path from $X_{\alpha,e,0}$ to $X_{\alpha,e,1}$ which contradicts the evaluation maps.

Theorem

The Borel Dual Ramsey Theorem for completely determined Borel colorings does not hold in HYP.

The idea of the proof is the same as the previous ones: we construct a coloring which successfully diagonalizes against all homogeneous solutions.

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The general pattern is that, in HYP, "choice like constructions" work for completely determined Borel sets.

Question

Is there a principle (weaker than ATR_0*) which suffices to rule out this defective behavior?*

For instance "there is no completely determined Borel well-ordering of sets" is something we would expect, and would at least complicate some of the other constructions here.