Computable Structure Theory of Continuous Logic

Caleb M.H. Camrud

Brown University (Iowa State University)
Organization

This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
2. Arithmetical hierarchy and computable presentations.
4. Computable infinitary continuous logic and hyperarithmetic numerals.
5. Diagram complexity of a computably presented metric structure.
6. Future work on r.i.c.e. relations in the continuous setting.
Organization

This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
2. Arithmetical hierarchy and computable presentations.
Organization

This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
2. Arithmetical hierarchy and computable presentations.
This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
2. Arithmetical hierarchy and computable presentations.
4. Computable infinitary continuous logic and hyperarithmetic numerals.
This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
2. Arithmetical hierarchy and computable presentations.
4. Computable infinitary continuous logic and hyperarithmetic numerals.
5. Diagram complexity of a computably presented metric structure.
This talk is organized as follows.

1. Continuous logic, metric structures, and presentations.
2. Arithmetical hierarchy and computable presentations.
4. Computable infinitary continuous logic and hyperarithmetic numerals.
5. Diagram complexity of a computably presented metric structure.
6. Future work on r.i.c.e. relations in the continuous setting.
In metric spaces, the metric encodes *more* information than simple equality.
Continuous logic

In metric spaces, the metric encodes \textit{more} information than simple equality.

\[ d(x, y) = 0 \text{ if and only if } x = y. \]
Continuous logic

In metric spaces, the metric encodes *more* information than simple equality.

\[ d(x, y) = 0 \text{ if and only if } x = y. \]

If \( d(x, y) < d(x, z) \), then \( y \) is *closer* to \( x \) than \( z \) is to \( x \).
Continuous logic

Model Theory for Metric Structures, Ben Yaacov et al. 2006
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov *et al.* 2006

Space of truth values is \([0, 1]\) instead of \(\{0, 1\}\).
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov *et al.* 2006

Space of truth values is $[0, 1]$ instead of $\{0, 1\}$.
- $d$ in the place of $\equiv$.
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov *et al.* 2006

Space of truth values is $[0, 1]$ instead of $\{0, 1\}$.

- $d$ in the place of $\equiv$.
- Continuous *predicates* instead of relations.
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov et al. 2006

Space of truth values is $[0, 1]$ instead of $\{0, 1\}$.

- $d$ in the place of $\equiv$.
- Continuous *predicates* instead of relations.
- Functions (operations) must be continuous.

Since $d(x, y) = 0$ if and only if $x = y$, “0” corresponds to truth, while “1” to falsity.
Continuous logic


Space of truth values is \([0, 1]\) instead of \([0, 1]\).

- \(d\) in the place of \(=\).
- Continuous *predicates* instead of relations.
- Functions (operations) must be continuous.
- \(\inf\) and \(\sup\) in the place of \(\exists\) and \(\forall\).
Continuous logic


Space of truth values is $[0, 1]$ instead of $\{0, 1\}$.

► $d$ in the place of $\equiv$.
► Continuous *predicates* instead of relations.
► Functions (operations) must be continuous.
► $\inf$ and $\sup$ in the place of $\exists$ and $\forall$.
► Still has negation $\neg$. 
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov *et al.* 2006

Space of truth values is $[0, 1]$ instead of $\{0, 1\}$.

- $d$ in the place of $\equiv$.
- Continuous *predicates* instead of relations.
- Functions (operations) must be continuous.
- $\inf$ and $\sup$ in the place of $\exists$ and $\forall$.
- Still has negation $\neg$.
- Has $\models$ instead of reverse implication ($\leftarrow$).
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov *et al.* 2006

Space of truth values is \([0, 1]\) instead of \(\{0, 1\}\).
- \(d\) in the place of \(=\).
- Continuous *predicates* instead of relations.
- Functions (operations) must be continuous.
- \(\inf\) and \(\sup\) in the place of \(\exists\) and \(\forall\).
- Still has negation \(\neg\).
- Has \(\dagger\) instead of reverse implication (\(\leftarrow\)).
- Includes an additional logical connective \(\frac{1}{2}\).
Continuous logic

*Model Theory for Metric Structures*, Ben Yaacov *et al.* 2006

Space of truth values is $[0, 1]$ instead of $\{0, 1\}$.
- $d$ in the place of $\equiv$.
- Continuous *predicates* instead of relations.
- Functions (operations) must be continuous.
- $\inf$ and $\sup$ in the place of $\exists$ and $\forall$.
- Still has negation $\neg$.
- Has $\not\rightarrow$ instead of reverse implication ($\not\leftarrow$).
- Includes an additional logical connective $\frac{1}{2}$.

Since $d(x, y) = 0$ if and only if $x = y$, “0” corresponds to truth, while “1” to falsity.
Continuous logic
Well-formed formulas (wffs) are defined in the standard way.
Continuous logic

Well-formed formulas (wffs) are defined in the standard way.

Quantifier-free formulas look like

\[ P(t_0, \ldots, t_\eta) \equiv \neg \phi_1 \lor \phi_2 \land \psi, \]
where \( \phi \) and \( \psi \) are also quantifier-free.
Continuous logic

Well-formed formulas (wffs) are defined in the standard way.

Quantifier-free formulas look like

$\forall$ $P(t_0, \ldots, t_{\eta(P)} - 1)$
Continuous logic

Well-formed formulas (wffs) are defined in the standard way.

Quantifier-free formulas look like

- $P(t_0, \ldots, t_{\eta(P)}-1)$
- $\neg \varphi$, $\frac{1}{2} \varphi$, and $\varphi \div \psi$, where $\varphi$ and $\psi$ are also quantifier-free.
Continuous logic

The $\Sigma_N$ and $\Pi_N$ wff’s are defined similarly to the classical case.
Continuous logic

The $\Sigma_N$ and $\Pi_N$ wff’s are defined similarly to the classical case.

For example, let $\varphi$ be quantifier-free.
The $\Sigma_N$ and $\Pi_N$ wff’s are defined similarly to the classical case.

For example, let $\varphi$ be quantifier-free.

$\sup_{x_0} \varphi$ is $\Pi_1$. 
Continuous logic

The $\Sigma_N$ and $\Pi_N$ wff’s are defined similarly to the classical case.

For example, let $\varphi$ be quantifier-free.

- $\sup_{x_0} \varphi$ is $\Pi_1$.
- $\inf_{x_1} \sup_{x_0} \varphi$ is $\Sigma_2$. 
Continuous logic

The $\Sigma_N$ and $\Pi_N$ wff’s are defined similarly to the classical case.

For example, let $\varphi$ be quantifier-free.

- $\sup_{x_0} \varphi$ is $\Pi_1$.
- $\inf_{x_1} \sup_{x_0} \varphi$ is $\Sigma_2$.
- $\sup_{x_{N-1}} \inf_{x_{N-2}} \ldots \varphi$ is $\Pi_N$.
### Continuous Logic

<table>
<thead>
<tr>
<th><strong>Shorthand</strong></th>
<th><strong>String</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi \lor \psi$</td>
<td>$\neg((\neg \varphi) \div \psi)$</td>
</tr>
<tr>
<td>$\varphi \land \psi$</td>
<td>$\varphi \div (\varphi \div \psi)$</td>
</tr>
<tr>
<td>$\varphi \leftrightarrow \psi$</td>
<td>$(\varphi \div \psi) \lor (\psi \div \varphi)$</td>
</tr>
<tr>
<td>0</td>
<td>sup$_x$ $d(x, x)$</td>
</tr>
<tr>
<td>1</td>
<td>$\neg 0$</td>
</tr>
<tr>
<td>$\varphi \vdash \psi$</td>
<td>$\neg((1 \div \varphi) \div \psi))$</td>
</tr>
<tr>
<td>$m \varphi$</td>
<td>$(\ldots(\varphi \vdash \varphi) \vdash \ldots \vdash \varphi)$</td>
</tr>
<tr>
<td>$2^{-k}$</td>
<td>$m$-many</td>
</tr>
</tbody>
</table>

Caleb M.H. Camrud
Brown University (Iowa State University)
Computable Structure Theory of Continuous Logic
Infinitary continuous logic

For a countable index set $I$, if $(\varphi_i)_{i \in I}$ share a tuple of free variables and are uniformly equicontinuous in those variables, then

$$\bigwedge_{i \in I} \varphi_i \quad \text{and} \quad \bigvee_{i \in I} \varphi_i$$

are infinitary formulas.
Metric structures

When $(|\mathcal{M}|, d)$ is a pseudometric space of diameter 1, $P^\mathcal{M} : |\mathcal{M}| \eta(P) \to [0, 1]$ are *predicates* (functionals),
Metric structures

When $(|M|, d)$ is a pseudometric space of diameter 1, $P^M : |M| \eta(P) \rightarrow [0, 1]$ are predicates (functionals),

$f^M : |M| \eta(f) \rightarrow |M|$ are functions (operations),
Metric structures

When \(|\mathcal{M}|, d\) is a pseudometric space of diameter 1, \(P^\mathcal{M} : |\mathcal{M}|^\eta(P) \to [0, 1]\) are predicates (functionals),

\(f^\mathcal{M} : |\mathcal{M}|^\eta(f) \to |\mathcal{M}|\) are functions (operations), and

\(c^\mathcal{M} \in |\mathcal{M}|\) are points,
Metric structures

When \(|M|, d\) is a pseudometric space of diameter 1,
\(\mathcal{P}^M : |M|^{\eta(P)} \to [0, 1]\) are *predicates* (functionals),

\(f^M : |M|^{\eta(f)} \to |M|\) are *functions* (operations), and

\(c^M \in |M|\) are *points*,

\[ M = (|M|, d, \{ P^M : P \in \mathcal{P} \}, \{ f^M : f \in \mathcal{F} \}, \{ c^M : c \in \mathcal{C} \}) \]

is called a *continuous L-pre-structure*. 
Metric structures

When $\langle |M|, d \rangle$ is a pseudometric space of diameter 1, $P^M : |M|^{\eta(P)} \to [0, 1]$ are predicates (functionals),

$f^M : |M|^{\eta(f)} \to |M|$ are functions (operations), and

$c^M \in |M|$ are points,

$M = (|M|, d, \{P^M : P \in \mathcal{P}\}, \{f^M : f \in \mathcal{F}\}, \{c^M : c \in \mathcal{C}\})$,

is called a continuous $L$-pre-structure.

If, moreover, $\langle |M|, d \rangle$ is a complete metric space, then $M$ is an $L$-structure.
Metric structures

The *interpretation* of sentences in an $L$-pre-structure $\mathcal{M}$ is then defined as follows.
Metric structures

The *interpretation* of sentences in an $L$-pre-structure $\mathcal{M}$ is then defined as follows.

\[
P(t_0, \ldots, t_{N-1})^\mathcal{M} := P^\mathcal{M}(t_0^\mathcal{M}, \ldots, t_{N-1}^\mathcal{M})
\]
The *interpretation* of sentences in an $L$-pre-structure $\mathcal{M}$ is then defined as follows.

\[
(P(t_0, \ldots, t_{N-1}))^\mathcal{M} := P^\mathcal{M}(t_0^\mathcal{M}, \ldots, t_{N-1}^\mathcal{M})
\]

\[
(\neg \varphi)^\mathcal{M} := 1 - \varphi^\mathcal{M}
\]
The *interpretation* of sentences in an $L$-pre-structure $\mathcal{M}$ is then defined as follows.

\[
(P(t_0, \ldots, t_{N-1}))^\mathcal{M} := P^\mathcal{M}(t_0^\mathcal{M}, \ldots, t_{N-1}^\mathcal{M})
\]

\[
(-\varphi)^\mathcal{M} := 1 - \varphi^\mathcal{M} \quad \left(\frac{1}{2}\varphi\right)^\mathcal{M} := \frac{1}{2} \varphi^\mathcal{M}
\]
The *interpretation* of sentences in an $L$-pre-structure $\mathcal{M}$ is then defined as follows.

$$
(P(t_0, \ldots, t_{N-1}))^\mathcal{M} := P^\mathcal{M}(t_0^\mathcal{M}, \ldots, t_{N-1}^\mathcal{M})
$$

$$
(\neg \varphi)^\mathcal{M} := 1 - \varphi^\mathcal{M} \quad (\frac{1}{2} \varphi)^\mathcal{M} := \frac{1}{2} \varphi^\mathcal{M}
$$

$$
(\varphi \cdot \psi)^\mathcal{M} := \max\{0, \varphi^\mathcal{M} - \psi^\mathcal{M}\}
$$
The interpretation of sentences in an \( L \)-pre-structure \( \mathcal{M} \) is then defined as follows.

\[
(P(t_0, \ldots, t_{N-1}))^\mathcal{M} := P^\mathcal{M}(t_0^\mathcal{M}, \ldots, t_{N-1}^\mathcal{M})
\]

\[
(\neg \varphi)^\mathcal{M} := 1 - \varphi^\mathcal{M} \quad (\frac{1}{2} \varphi)^\mathcal{M} := \frac{1}{2} \varphi^\mathcal{M}
\]

\[
(\varphi \cdot \psi)^\mathcal{M} := \max\{0, \varphi^\mathcal{M} - \psi^\mathcal{M}\}
\]

\[
(\sup_x \varphi(x))^\mathcal{M} := \sup_{a \in |\mathcal{M}|} \varphi^\mathcal{M}(a)
\]
Metric structures

The interpretation of sentences in an $L$-pre-structure $\mathcal{M}$ is then defined as follows.

$$(P(t_0, \ldots, t_{N-1}))^\mathcal{M} := P^\mathcal{M}(t_0^\mathcal{M}, \ldots, t_{N-1}^\mathcal{M})$$

$$(\neg \varphi)^\mathcal{M} := 1 - \varphi^\mathcal{M} \quad (\frac{1}{2} \varphi)^\mathcal{M} := \frac{1}{2} \varphi^\mathcal{M}$$

$$(\varphi \cdot \psi)^\mathcal{M} := \max\{0, \varphi^\mathcal{M} - \psi^\mathcal{M}\}$$

$$(\sup_x \varphi(x))^\mathcal{M} := \sup_{a \in |\mathcal{M}|} \varphi^\mathcal{M}(a) \quad (\inf_x \varphi(x))^\mathcal{M} := \inf_{a \in |\mathcal{M}|} \varphi^\mathcal{M}(a)$$
Metric structures

Moreover, the *interpretation* of infinitary formulas sentences is as follows.
Moreover, the interpretation of infinitary formulas sentences is as follows.

\[
\left( \bigwedge_{i \in I} \varphi_i \right)^m := \inf_{i \in I} \varphi_i^m
\]
Metric structures

Moreover, the interpretation of infinitary formulas sentences is as follows.

\[
\left( \bigwedge_{i \in I} \varphi_i \right)^m := \inf_{i \in I} \varphi_i^m
\]

\[
\left( \bigvee_{i \in I} \varphi_i \right)^m := \sup_{i \in I} \varphi_i^m
\]
Metric structures

\( \mathcal{M} \) satsifies (or models) \( \varphi \) if \( \varphi^\mathcal{M} = 0 \).
Metric structures

\( M \) satsifies (or models) \( \varphi \) if \( \varphi^M = 0 \). (\( M \models \varphi \))

When \( \Gamma \) is a set of wffs, \( M \models \Gamma \) means \( M \models \varphi \) for every \( \varphi \in \Gamma \).
Metric structures

$M$ satisfies (or models) $\varphi$ if $\varphi^M = 0$. ($M \models \varphi$)

When $\Gamma$ is a set of wffs, $M \models \Gamma$ means $M \models \varphi$ for every $\varphi \in \Gamma$. 
Example

The unit ball of a Banach space over $\mathbb{R}$, the metric induced by the norm, as functions all binary maps of the form

$$f_{\alpha,\beta}(x, y) = \alpha x + \beta y$$

where $|\alpha| + |\beta| \leq 1$ as scalars, and the additive identity 0 and some choice of normal basis $(e_i)_{i \in I}$ as distinguished points.
Example
The unit ball of a Banach space over $\mathbb{R}$, the metric induced by the norm, as functions all binary maps of the form

$$f_{\alpha,\beta}(x, y) = \alpha x + \beta y$$

where $|\alpha| + |\beta| \leq 1$ as scalars, and the additive identity 0 and some choice of normal basis $(e_i)_{i \in I}$ as distinguished points.

$$ \left( d\left( f_{\frac{3}{4}, 0}(e_0, e_3), 0 \right) \right)^m = $$
Metric structures

Example

The unit ball of a Banach space over $\mathbb{R}$, the metric induced by the norm, as functions all binary maps of the form

$$f_{\alpha,\beta}(x, y) = \alpha x + \beta y$$

where $|\alpha| + |\beta| \leq 1$ as scalars, and the additive identity 0 and some choice of normal basis $(e_i)_{i \in I}$ as distinguished points.

$$\left( d\left( \frac{3}{4}, 0, (e_0, e_3), 0 \right) \right)^m = \| \left( \frac{3}{4} \cdot e_0 + 0 \cdot e_3 \right) - 0 \|= \phantom{\)}$$
Metric structures

Example

The unit ball of a Banach space over $\mathbb{R}$, the metric induced by the norm, as functions all binary maps of the form

$$f_{\alpha,\beta}(x, y) = \alpha x + \beta y$$

where $|\alpha| + |\beta| \leq 1$ as scalars, and the additive identity 0 and some choice of normal basis $(e_i)_{i \in I}$ as distinguished points.

$$(d(f_{\frac{3}{4},0}(e_0, e_3), 0))^m = \| (\frac{3}{4} \cdot e_0 + 0 \cdot e_3) - 0 \| = \frac{3}{4}.$$
Metric structures

*N.B.* Any classical structure can be made a metric structure by applying the discrete metric.
Metric structures

N.B. Any classical structure can be made a metric structure by applying the discrete metric.

In this case, the metric just serves to indicate equality.
Presentations

Given a structure $\mathcal{M}$ and $A \subseteq |\mathcal{M}|$, we define the \textit{algebra generated by} $A$ to be the smallest subset of $|\mathcal{M}|$ containing $A$ that is closed under every function of $\mathcal{M}$.
Presentations

Given a structure $\mathcal{M}$ and $A \subseteq |\mathcal{M}|$, we define the algebra generated by $A$ to be the smallest subset of $|\mathcal{M}|$ containing $A$ that is closed under every function of $\mathcal{M}$.

A pair $(\mathcal{M}, g)$ is called a presentation of $\mathcal{M}$ if $g : \mathbb{N} \to |\mathcal{M}|$ is a map such that the algebra generated by $\text{ran}(g)$ is dense.
Given a presentation $M^\# = (M, g)$, every $a \in \text{ran}(g)$ is called a distinguished point of $M^\#$. 

Each point in the algebra generated by the distinguished points is called a rational point of $M^\#$. 

Caleb M.H. Camrud
Brown University (Iowa State University)
Computable Structure Theory of Continuous Logic
Given a presentation $\mathcal{M}^\# = (\mathcal{M}, g)$, every $a \in \text{ran}(g)$ is called a *distinguished point* of $\mathcal{M}^\#$.

Each point in the algebra generated by the distinguished points is called a *rational point* of $\mathcal{M}^\#$. 

Caleb M.H. Camrud
Brown University (Iowa State University)
Computable Structure Theory of Continuous Logic
The Arithmetical Hierarchy

Definition
The $\Sigma^0_n$, $\Pi^0_n$, and $\Delta^0_n$ sets are defined recursively for every $n \in \mathbb{N} \setminus \{0\}$. 
The Arithmetical Hierarchy

Definition
The $\Sigma^0_n$, $\Pi^0_n$, and $\Delta^0_n$ sets are defined recursively for every $n \in \mathbb{N} \setminus \{0\}$. A set $A \subseteq \mathbb{N}$ is

- $\Sigma^0_1$ if there is some computable binary relation $R \subseteq \mathbb{N}^2$ such that
  
  $$k \in A \iff \exists s \in \mathbb{N} \ R(s, k);$$

- $\Pi^0_1$ if there is some computable binary relation $R \subseteq \mathbb{N}^2$ such that
  
  $$k \in A \iff \forall s \in \mathbb{N} \ R(s, k);$$
The Arithmetical Hierarchy

Definition
The $\Sigma^0_n$, $\Pi^0_n$, and $\Delta^0_n$ sets are defined recursively for every $n \in \mathbb{N} \setminus \{0\}$. A set $A \subseteq \mathbb{N}$ is

- $\Sigma^0_1$ if there is some computable binary relation $R \subseteq \mathbb{N}^2$ such that
  
  $$k \in A \iff \exists s \in \mathbb{N} \ R(s, k);$$

- $\Pi^0_1$ if there is some computable binary relation $R \subseteq \mathbb{N}^2$ such that
  
  $$k \in A \iff \forall s \in \mathbb{N} \ R(s, k);$$
The Arithmetical Hierarchy

- $\Sigma^0_n$ if there is some $\Pi^0_{n-1}$ binary relation $R \subseteq \mathbb{N}^2$ such that

  \[ k \in A \iff \exists s \in \mathbb{N} \ R(s, k); \]
The Arithmetical Hierarchy

- **$\Sigma^0_n$** if there is some $\Pi^0_{n-1}$ binary relation $R \subseteq \mathbb{N}^2$ such that
  \[ k \in A \iff \exists s \in \mathbb{N} \ R(s, k); \]

- **$\Pi^0_n$** if there is some $\Sigma^0_{n-1}$ binary relation $R \subseteq \mathbb{N}^2$ such that
  \[ k \in A \iff \forall s \in \mathbb{N} \ R(s, k); \]
The Arithmetical Hierarchy

- $\Sigma^0_n$ if there is some $\Pi^0_{n-1}$ binary relation $R \subseteq \mathbb{N}^2$ such that
  $$k \in A \iff \exists s \in \mathbb{N} \ R(s, k);$$

- $\Pi^0_n$ if there is some $\Sigma^0_{n-1}$ binary relation $R \subseteq \mathbb{N}^2$ such that
  $$k \in A \iff \forall s \in \mathbb{N} \ R(s, k);$$

- $\Delta^0_n$ if it is both $\Sigma^0_n$ and $\Pi^0_n$. 
A set $A \subseteq \mathbb{N}$ is \textit{arithmetical} if it is $\Sigma^0_n$ for some $n \in \mathbb{N}$.
A set $A \subseteq \mathbb{N}$ is *arithmetical* if it is $\Sigma^n_0$ for some $n \in \mathbb{N}$.

There is a natural way of extending each of the above classes to $\Sigma_\alpha^0$, $\Pi_\alpha^0$, and $\Delta_\alpha^0$ for every computable ordinal $\alpha$. 
A set $A \subseteq \mathbb{N}$ is *arithmetical* if it is $\Sigma^0_n$ for some $n \in \mathbb{N}$.

There is a natural way of extending each of the above classes to $\Sigma^0_\alpha$, $\Pi^0_\alpha$, and $\Delta^0_\alpha$ for every computable ordinal $\alpha$.

A set $A \subseteq \mathbb{N}$ is *hyperarithmetical* if it is $\Sigma^0_\alpha$ for some computable ordinal $\alpha$. 
Computable presentations

Definition
Let $A$ be a countable set. A map $f : A \rightarrow \mathbb{R}$ is *computable* if there is an effective procedure which, given $a \in A$ and $k \in \mathbb{N}$, outputs a rational $q$ such that

$$|f(a) - q| < 2^{-k}.$$
Computable presentations

Definition
Let $A$ be a countable set. A map $f : A \to \mathbb{R}$ is computable if there is an effective procedure which, given $a \in A$ and $k \in \mathbb{N}$, outputs a rational $q$ such that

$$|f(a) - q| < 2^{-k}.$$ 

Definition
A presentation $\mathcal{M}^\#$ is computable if the predicates of $\mathcal{M}$ are uniformly computable on the rational points of $\mathcal{M}^\#$. 
Computable presentations

Definition
Let $A$ be a countable set. A map $f : A \rightarrow \mathbb{R}$ is computable if there is an effective procedure which, given $a \in A$ and $k \in \mathbb{N}$, outputs a rational $q$ such that

$$|f(a) - q| < 2^{-k}.$$

Definition
A presentation $\mathcal{M}^#$ is computable if the predicates of $\mathcal{M}$ are uniformly computable on the rational points of $\mathcal{M}^#$. We say that a metric structure is computably presentable if it has a computable presentation.
Motivation for First Result

Motivation for First Result


**Theorem (Effective Completeness)**

*In classical logic, if a theory is decidable (meaning its set of consequences is computable), then it is modeled by a computable structure.*
Definition (Ben Yaacov and Pedersen, 2010)

Let $\Gamma$ be a set of wffs. The *degree of truth with respect to* $\Gamma$ ($\cdot \circ \Gamma$) is a map from wffs to $[0, 1]$, defined as

$$\varphi^\circ \Gamma := \sup \{ \varphi^M : M \models \Gamma \}.$$
Generalized Effective Completeness

Definition (Ben Yaacov and Pedersen, 2010)
Let $\Gamma$ be a set of wffs. The degree of truth with respect to $\Gamma$ ($\cdot\Gamma$) is a map from wffs to $[0, 1]$, defined as

$$\varphi_\Gamma := \sup \{ \varphi^\mathcal{M} : \mathcal{M} \models \Gamma \}.$$ 

Definition (Ben Yaacov and Pedersen, 2010)
A theory $T$ is decidable if $\cdot_T$ is a computable map from $\langle \varphi_n \rangle_{n \in \mathbb{N}}$ into $[0, 1]$. 
Generalized Effective Completeness

Definition
Given a theory $T$, we say that $X \in \mathbb{N}^\mathbb{N}$ is a name of $T$ if the following hold.

1. For every $n, k \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $\langle n, k, m \rangle \in \text{ran}(X)$.
2. For every $n, k, m \in \mathbb{N}$, if $\langle n, k, m \rangle \in \text{ran}(X)$, then $\varphi_m \in (\varphi_n^T - 2^{\varphi_T + 2^k})$. 

Proposition
A theory is decidable if and only if it has a computable name.
Generalized Effective Completeness

Definition
Given a theory $T$, we say that $X \in \mathbb{N}^\mathbb{N}$ is a *name* of $T$ if the following hold.

- For every $n, k \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $\langle n, k, m \rangle \in \text{ran}(X)$.

Proposition
A theory is decidable if and only if it has a computable name.
Definition
Given a theory $T$, we say that $X \in \mathbb{N}^\mathbb{N}$ is a *name* of $T$ if the following hold.

- For every $n, k \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $\langle n, k, m \rangle \in \text{ran}(X)$.
- For every $n, k, m \in \mathbb{N}$, if $\langle n, k, m \rangle \in \text{ran}(X)$, then $q_m \in \left[ (\varphi_n)_T - 2^{-k}, (\varphi_n)_T + 2^{-k} \right]$. 

Proposition
A theory is decidable if and only if it has a computable name.
Generalized Effective Completeness

Definition
Given a theory $T$, we say that $X \in \mathbb{N}^{\mathbb{N}}$ is a name of $T$ if the following hold.

- For every $n, k \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that $\langle n, k, m \rangle \in \text{ran}(X)$.
- For every $n, k, m \in \mathbb{N}$, if $\langle n, k, m \rangle \in \text{ran}(X)$, then $q_m \in \left[ (\varphi_n)_T - 2^{-k}, (\varphi_n)_T + 2^{-k} \right]$.

Proposition
A theory is decidable if and only if it has a computable name.
Generalized Effective Completeness

**Lemma**

There is an effective procedure which given $X$, a name of an $L$-theory $T$, outputs $\Phi(X) \subseteq \mathbb{N}$ such that $T \cup \{\theta_n : n \in \Phi(X)\}$ is consistent, and for every pair of wffs $\varphi$ and $\psi$, either $\varphi$ is provably equivalent to $\psi$, or exactly one of $\varphi \vdash \psi$ or $\psi \vdash \varphi$ is in $\{\theta_n : n \in \Phi(X)\}$. 
Generalized Effective Completeness

Theorem (Generalized Effective Completeness)

There is an effective procedure which, given \( X \), a name of an \( L \)-theory \( T \), produces a presentation of a structure \( M \) such that \( M \models T \).
Generalized Effective Completeness

Theorem (Generalized Effective Completeness)

There is an effective procedure which, given $X$, a name of an $L$-theory $T$, produces a presentation of a structure $M$ such that $M \vDash T$.

Corollary (Effective Completeness of Continuous Logic)

Every decidable theory is modeled by a computably presentable structure.
Motivation for Second Result

Ben Yaacov and Pedersen (2010) noted that for any dyadic $r \in [0, 1]$ (i.e., number of the form $\frac{\ell}{2^k}$), there is a finitary sentence which is universally interpreted as $r$. 
Motivation for Second Result

Ben Yaacov and Pedersen (2010) noted that for any dyadic $r \in [0, 1]$ (i.e., number of the form $\frac{\ell}{2^k}$), there is a finitary sentence which is universally interpreted as $r$. (Recall the shorthands given previously.)
Ben Yaacov and Pedersen (2010) noted that for any dyadic \( r \in [0, 1] \) \((i.e., \) number of the form \( \frac{\ell}{2^k} \)), there is a finitary sentence which is universally interpreted as \( r \). (Recall the shorthands given previously.)

- \( 0 := \sup_x d(x, x) \).
- \( 1 := \neg 0 \).
- \( \frac{1}{2^k} := \frac{1}{2} \cdots \frac{1}{2} 1 \). (\( k \)-many \( \frac{1}{2} \) connectives)
- \( \frac{\ell}{2^k} := \neg (1 \div \frac{1}{2^k} \div \cdots \div \frac{1}{2^k}) \). (\( \ell \)-many \( \div \frac{1}{2^k} \) terms)
Motivation for Second Result

Ben Yaacov and Pedersen (2010) noted that for any dyadic $r \in [0, 1]$ \textit{(i.e., number of the form $\frac{\ell}{2^k}$)}, there is a finitary sentence which is universally interpreted as $r$. (Recall the shorthands given previously.)

- $0 := \sup_x d(x, x)$.
- $1 := \neg 0$.
- $\frac{1}{2^k} := \frac{1}{2} \ldots \frac{1}{2} 1$. ($k$-many $\frac{1}{2}$ connectives)
- $\frac{\ell}{2^k} := \neg (1 \div \frac{1}{2^k} \div \ldots \div \frac{1}{2^k})$. ($\ell$-many $\div \frac{1}{2^k}$ terms)

But what if we consider computable infinitary formulas?
Computable Infinitary Formulas

Heuristic

The computable infinitary wffs are heuristically given as follows, where \( \alpha \) is a computable ordinal.
Computable Infinitary Formulas

Heuristic

The *computable infinitary wffs* are heuristically given as follows, where $\alpha$ is a computable ordinal.

- The $\Sigma^c_0 = \Pi^c_0$ sets include all quantifier-free, finitary wffs.
**Computable Infinitary Formulas**

**Heuristic**

The *computable infinitary wffs* are heuristically given as follows, where $\alpha$ is a computable ordinal.

- The $\Sigma^c_0 = \Pi^c_0$ sets include all quantifier-free, finitary wffs.
- A wff $\varphi$ is $\Sigma^c_\alpha$ if it is of the form

$$\varphi = \bigwedge_{i \in I} \inf_x \psi_i$$

where $I \subseteq \mathbb{N}$ is c.e., each $\psi_i \in \Pi^c_\beta$, for some $\beta < \alpha$, and a modulus of continuity for $\varphi$ exists that is computable from a code for $\varphi$. 
A wff $\varphi$ is $\Pi^c_\alpha$ if it is of the form

$$\varphi = \bigwedge_{i \in I} \sup_{x} \psi_i$$

where $I \subseteq \mathbb{N}$ is c.e., each $\psi_i \in \Sigma^c_\beta$, for some $\beta < \alpha$, and a modulus of continuity for $\varphi$ exists that is computable from a code for $\varphi$. 
Hyperarithmetical Numerals

**Theorem (C., McNicholl)**

*There is an effective procedure which, given a hyperarithmetical right Dedekind cut of a real number \( r \in [0,1] \), outputs a computable infinitary sentence such that the following hold.*

1. If the right Dedekind cut given is \( \Pi^0_\alpha \), then the output is a \( \Pi^c_\alpha \) sentence \( \phi \) such that for every structure \( M \), \( \phi_M = r \).
2. If the right Dedekind cut given is \( \Sigma^0_\alpha \), then the output is a \( \Sigma^c_\alpha \) sentence \( \phi \) such that for every structure \( M \), \( \phi_M = r \).
Theorem (C., McNicholl)

There is an effective procedure which, given a hyperarithmetical right Dedekind cut of a real number $r \in [0, 1]$, outputs a computable infinitary sentence such that the following hold.

- If the right Dedekind cut given is $\Pi_\alpha^0$, then the output is a $\Pi^c_\alpha$ sentence $\varphi$ such that for every structure $\mathcal{M}$, $\varphi^\mathcal{M} = r$. 
Hyperarithmetical Numerals

Theorem (C., McNicholl)

There is an effective procedure which, given a hyperarithmetical right Dedekind cut of a real number $r \in [0, 1]$, outputs a computable infinitary sentence such that the following hold.

- If the right Dedekind cut given is $\Pi^0_\alpha$, then the output is a $\Pi^c_\alpha$ sentence $\varphi$ such that for every structure $\mathcal{M}$, $\varphi^\mathcal{M} = r$.
- If the right Dedekind cut given is $\Sigma^0_\alpha$, then the output is a $\Sigma^c_\alpha$ sentence $\varphi$ such that for every structure $\mathcal{M}$, $\varphi^\mathcal{M} = r$. 
Discussion of Second Result

In other words, for any nonzero computable ordinal $\alpha$ and any right $\Pi^0_\alpha$ (or $\Sigma^0_\alpha$) real number $r \in [0, 1]$, there is a $\Pi^c_\alpha$ (respectively, $\Sigma^c_\alpha$) sentence $\varphi$ such that for every metric structure $\mathcal{M}$, $\varphi^\mathcal{M} = r$. 
Discussion of Second Result

In other words, for any nonzero computable ordinal $\alpha$ and any right $\Pi^0_\alpha$ (or $\Sigma^0_\alpha$) real number $r \in [0, 1]$, there is a $\Pi^c_\alpha$ (respectively, $\Sigma^c_\alpha$) sentence $\varphi$ such that for every metric structure $\mathcal{M}$, $\varphi^\mathcal{M} = r$.

These are numerals!
Discussion of Second Result

In other words, for any nonzero computable ordinal $\alpha$ and any right $\Pi^0_\alpha$ (or $\Sigma^0_\alpha$) real number $r \in [0, 1]$, there is a $\Pi^c_\alpha$ (respectively, $\Sigma^c_\alpha$) sentence $\varphi$ such that for every metric structure $M$, $\varphi^M = r$.

These are numerals!

**Corollary (C., McNicholl)**

*Suppose $M$ is an interpretation of $L^c_{\omega_1 \omega}$, and suppose $X$ computes the continuous theory of $M$. Then, $X$ computes every hyperarithmetic set. Thus, no hyperarithmetic set computes the continuous theory of any metric structure.*
Discussion of Second Result

We proved this by effective transfinite recursion on notations of computable ordinals.
Discussion of Second Result

We proved this by effective transfinite recursion on notations of computable ordinals. Along the way, we provided the following new definition and lemma.
Discussion of Second Result

We proved this by effective transfinite recursion on notations of computable ordinals. Along the way, we provided the following new definition and lemma.

_N.B._ Though slightly unconventional, for \( s \in \mathbb{R} \), by “a right Dedekind cut of \( s \)”, we mean either \( D^>(s) \) or \( D^\geq(s) \).

Caleb M.H. Camrud
Brown University (Iowa State University)
Computable Structure Theory of Continuous Logic
Discussion of Second Result

We proved this by effective transfinite recursion on notations of computable ordinals. Along the way, we provided the following new definition and lemma.

*N.B.* Though slightly unconventional, for $s \in \mathbb{R}$, by “a right Dedekind cut of $s$”, we mean either $D^>(s)$ or $D^\geq(s)$.

**Definition**

Let $(r_n)_{n \in \omega}$ be a sequence of real numbers and $g : \omega \to \omega_1^{\text{CK}}$. We say $(r_n)_{n \in \omega}$ is **weakly uniformly right** $\Sigma_g^0 (\Pi_g^0)$ if there is a computable function $f : \omega \to \omega$ such that for all $n \in \omega$, $f(n)$ is a $\Sigma_{g(n)}^0 (\Pi_{g(n)}^0)$ index of a right Dedekind cut of $r_n$. 
Discussion of Second Result

Lemma (C., McNicholl)

Let $\alpha \in \omega_1^{CK}$ and $s \in \mathbb{R}$. Then the following hold uniformly.

1. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Sigma_0^0$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi_0^\beta$.

2. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Pi_0^\alpha$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma_0^\beta$.

3. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Sigma_0^\alpha$, then there is a computable map $h: \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi_0^h$.

4. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Pi_0^\alpha$, then there is a computable map $h: \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma_0^h$. 
Discussion of Second Result

Lemma (C., McNicholl)

Let $\alpha \in \omega_1^{CK}$ and $s \in \mathbb{R}$. Then the following hold uniformly.

1. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Sigma_\alpha^0$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi_\beta^0$.

2. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Pi_\alpha^0$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma_\beta^0$.

3. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Sigma_\alpha^0$, then there is a computable map $h : \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi_0^\alpha$.

4. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Pi_\alpha^0$, then there is a computable map $h : \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma_0^\alpha$. 
Lemma (C., McNicholl)

Let $\alpha \in \omega_1^{CK}$ and $s \in \mathbb{R}$. Then the following hold uniformly.

1. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Sigma^0_\alpha$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi^0_\beta$.

2. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Pi^0_\alpha$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma^0_\beta$. 
Discussion of Second Result

Lemma (C., McNicholl)

Let $\alpha \in \omega_1^{CK}$ and $s \in \mathbb{R}$. Then the following hold uniformly.

1. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Sigma_\alpha^0$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi^0_\beta$.

2. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Pi_\alpha^0$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma^0_\beta$.

3. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Sigma^0_\alpha$, then there is a computable map $h : \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi^0_h$. 
Lemma (C., McNicholl)

Let $\alpha \in \omega_1^{CK}$ and $s \in \mathbb{R}$. Then the following hold uniformly.

1. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Sigma_0^\alpha$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi_0^\beta$.

2. If $\alpha = \beta + 1$ and a right Dedekind cut of $s$ is $\Pi_0^\alpha$, then there is a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma_0^\beta$.

3. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Sigma_0^\alpha$, then there is a computable map $h : \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \inf_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Pi_0^h$.

4. If $\alpha$ is a limit ordinal and a right Dedekind cut of $s$ is $\Pi_0^\alpha$, then there is a computable map $h : \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n \in \omega}$ such that $s = \sup_{n \in \omega} r_n$ and $(r_n)_{n \in \omega}$ is weakly uniformly right $\Sigma_0^h$. 
Motivation for Final Results

We are then led to our final motivating question:
Motivation for Final Results

We are then led to our final motivating question:

*If a metric structure is computably presentable, how complex is the continuous theory of that structure?*
Motivation for Final Results

We are then led to our final motivating question:

*If a metric structure is computably presentable, how complex is the continuous theory of that structure?*

That is, given a computable presentation and a $\Sigma_N$ (or $\Pi_N$) sentence in the language of continuous logic, how hard is it for a computer to determine the truth-value of that sentence?
Motivation for Final Results

We are then led to our final motivating question:

*If a metric structure is computably presentable, how complex is the continuous theory of that structure?*

That is, given a computable presentation and a $\Sigma_N$ (or $\Pi_N$) sentence in the language of continuous logic, how hard is it for a computer to determine the truth-value of that sentence?

Similarly, we can also ask the same question about *computable infinitary* sentences in continuous logic.
Another way of phrasing this question is in terms of diagrams.
Diagram Complexity

Another way of phrasing this question is in terms of *diagrams*.

A diagram of structure basically just describes the truth value of every sentence of a given complexity.
Another way of phrasing this question is in terms of *diagrams*.

A diagram of structure basically just describes the truth value of every sentence of a given complexity.

*E.g.* The classical $\Sigma_1$ diagram of a structure describes the truth or falsity of every $\Sigma_1$ sentence.
Diagram Complexity

As the truth value of a sentence of continuous logic may be any real in $[0, 1]$, we introduce two kinds of diagrams at each level.
Diagram Complexity

As the truth value of a sentence of continuous logic may be any real in $[0, 1]$, we introduce two kinds of diagrams at each level.

The closed $\Sigma_N$ diagram is

$$\{(\varphi, r) : \varphi \in \Sigma_N \text{ and } \varphi^m \leq r\}.$$
As the truth value of a sentence of continuous logic may be any real in $[0, 1]$, we introduce two kinds of diagrams at each level.

The \textit{closed} $\Sigma_N$ diagram is

$$\{ (\varphi, r) : \varphi \in \Sigma_N \text{ and } \varphi^m \leq r \}.$$ 

The \textit{open} $\Sigma_N$ diagram is

$$\{ (\varphi, r) : \varphi \in \Sigma_N \text{ and } \varphi^m < r \}.$$
Diagram Complexity

As the truth value of a sentence of continuous logic may be any real in $[0, 1]$, we introduce two kinds of diagrams at each level.

The closed $\Sigma_N$ diagram is

$$\{(\varphi, r) : \varphi \in \Sigma_N \text{ and } \varphi^m \leq r\}.$$ 

The open $\Sigma_N$ diagram is

$$\{(\varphi, r) : \varphi \in \Sigma_N \text{ and } \varphi^m < r\}.$$ 

The $\Pi_N$ diagrams relativize.
The problem of uniformly deciding if one computable real number is less than another is $\Sigma^0_1$-complete.
Intuition

The problem of uniformly deciding if one computable real number is less than another is $\Sigma^0_1$-complete.

Why?
Intuition

The problem of uniformly deciding if one computable real number is less than another is $\Sigma_1^0$-complete.

Why? Computably check the $n$th digit of each number until you find a difference.
The problem of uniformly deciding if one computable real number is less than another is $\Sigma^0_1$-complete.

**Why?** Computably check the $n$th digit of each number until you find a difference.

The problem of uniformly deciding if one computable real number is less than or equal to another is $\Pi^0_1$-complete.
The problem of uniformly deciding if one computable real number is less than another is $\Sigma^0_1$-complete.

Why? Computably check the $n$th digit of each number until you find a difference.

The problem of uniformly deciding if one computable real number is less than or equal to another is $\Pi^0_1$-complete.

Why?
The problem of uniformly deciding if one computable real number is less than another is $\Sigma^0_1$-complete.

*Why?* Computably check the $n$th digit of each number until you find a difference.

The problem of uniformly deciding if one computable real number is less than or equal to another is $\Pi^0_1$-complete.

*Why?* To check equality, you would need to computably check every digit of each number to guarantee they all match.
Recall the classical result:
Recall the classical result:

\[ \Sigma_N, \{0,1\} \rightarrow \Sigma^0_N \]
Intuition

Recall the classical result:

\[ \Sigma_N, \{0, 1\} \rightarrow \Sigma^0_N \]

\[ \Pi_N, \{0, 1\} \rightarrow \Pi^0_N \]
In the continuous case:
Intuition

In the continuous case:

\[ \Sigma_N, <, [0, 1] \]
Intuition

*In the continuous case:*

\[ \Sigma_N, <, [0, 1] \rightarrow \Sigma_1^0 \Sigma_N^0 \]
Intuition

In the continuous case:

\[ \Sigma_N, \cdot, [0, 1] \rightarrow \Sigma^0 \Sigma^0_N \rightarrow \Sigma^0_N \]
In the continuous case:

\[ \Sigma_N, <, [0, 1] \quad \rightarrow \quad \Sigma^0_1 \Sigma^0_N \quad \rightarrow \quad \Sigma^0_N \]

\[ \Sigma_N, \leq, [0, 1] \]
Intuition

*In the continuous case:*

\[ \Sigma_N, <, [0, 1] \rightarrow \Sigma_1^0 \Sigma_N^0 \rightarrow \Sigma_N^0 \]

\[ \Sigma_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Sigma_N^0 \]
In the continuous case:

\[ \Sigma_N, \ <, \ [0,1] \quad \rightarrow \quad \Sigma^1_0 \Sigma^0_N \quad \rightarrow \quad \Sigma^0_N \]

\[ \Sigma_N, \ \leq, \ [0,1] \quad \rightarrow \quad \Pi^0_1 \Sigma^0_N \quad \rightarrow \quad \Pi^0_{N+1} \]
Intuition

In the continuous case:

\[
\Sigma_N, <, [0, 1] \quad \rightarrow \quad \Sigma_1^0 \Sigma_N^0 \quad \rightarrow \quad \Sigma_N^0
\]

\[
\Sigma_N, \leq, [0, 1] \quad \rightarrow \quad \Pi_1^0 \Sigma_N^0 \quad \rightarrow \quad \Pi_N^0_{N+1}
\]

\[
\Pi_N, <, [0, 1]
\]
In the continuous case:

\[ \Sigma_N, \prec, [0, 1] \rightarrow \Sigma_1^0 \Sigma_N^0 \rightarrow \Sigma_N^0 \]
\[ \Sigma_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Sigma_N^0 \rightarrow \Pi_{N+1}^0 \]
\[ \Pi_N, \prec, [0, 1] \rightarrow \Sigma_1^0 \Pi_N^0 \]
Intuition

In the continuous case:

\[
\Sigma_N, <, [0, 1] \rightarrow \Sigma_1^0 \Sigma_N^0 \rightarrow \Sigma_N^0
\]

\[
\Sigma_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Sigma_N^0 \rightarrow \Pi_{N+1}^0
\]

\[
\Pi_N, <, [0, 1] \rightarrow \Sigma_1^0 \Pi_N^0 \rightarrow \Sigma_{N+1}^0
\]
In the continuous case:

\[
\begin{align*}
\Sigma_N, <, [0, 1] & \rightarrow \Sigma^0 \Sigma^0_N \rightarrow \Sigma^0_N \\
\Sigma_N, \leq, [0, 1] & \rightarrow \Pi^0 \Sigma^0_N \rightarrow \Pi^0_{N+1} \\
\Pi_N, <, [0, 1] & \rightarrow \Sigma^0 \Pi^0_N \rightarrow \Sigma^0_{N+1} \\
\Pi_N, \leq, [0, 1] & \\
\end{align*}
\]
Intuition

In the continuous case:

\[ \Sigma_N, <, [0, 1] \rightarrow \Sigma_1^0 \Sigma_N \rightarrow \Sigma_N^0 \]

\[ \Sigma_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Sigma_N \rightarrow \Pi_{N+1}^0 \]

\[ \Pi_N, <, [0, 1] \rightarrow \Sigma_1^0 \Pi_N \rightarrow \Sigma_{N+1}^0 \]

\[ \Pi_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Pi_N \]
Intuition

*In the continuous case:*

\[
\Sigma_N, \prec, [0, 1] \rightarrow \Sigma_1^0 \Sigma_N^0 \rightarrow \Sigma_N^0
\]

\[
\Sigma_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Sigma_N^0 \rightarrow \Pi_{N+1}^0
\]

\[
\Pi_N, \prec, [0, 1] \rightarrow \Sigma_1^0 \Pi_N^0 \rightarrow \Sigma_{N+1}^0
\]

\[
\Pi_N, \leq, [0, 1] \rightarrow \Pi_1^0 \Pi_N^0 \rightarrow \Pi_N^0
\]
Summary of Finitary Result(s)

Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure, and let $N$ be a positive integer.

1. The closed quantifier-free diagram of $\mathcal{M}$ is $\Pi_{0}^{1}$, and the open quantifier-free diagram of $\mathcal{M}$ is $\Sigma_{0}^{1}$.

2. The closed $\Pi_{N}$ diagram of $\mathcal{M}$ is $\Pi_{0}^{N}$, and the open $\Pi_{N}$ diagram of $\mathcal{M}$ is $\Sigma_{0}^{N+1}$.

3. The closed $\Sigma_{N}$ diagram of $\mathcal{M}$ is $\Pi_{0}^{N+1}$, and the open $\Sigma_{N}$ diagram of $\mathcal{M}$ is $\Sigma_{0}^{N}$. 

Caleb M.H. Camrud
Brown University (Iowa State University)
Computable Structure Theory of Continuous Logic
Summary of Finitary Result(s)

Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure, and let $N$ be a positive integer.

1. The closed quantifier-free diagram of $\mathcal{M}$ is $\Pi_0^1$, and the open quantifier-free diagram of $\mathcal{M}$ is $\Sigma_1^0$. 
Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure, and let $N$ be a positive integer.

1. The closed quantifier-free diagram of $\mathcal{M}$ is $\Pi^0_1$, and the open quantifier-free diagram of $\mathcal{M}$ is $\Sigma^0_1$.

2. The closed $\Pi^0_N$ diagram of $\mathcal{M}$ is $\Pi^0_0$, and the open $\Pi^0_N$ diagram of $\mathcal{M}$ is $\Sigma^0_{N+1}$.
Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure, and let $N$ be a positive integer.

1. The closed quantifier-free diagram of $\mathcal{M}$ is $\Pi^0_1$, and the open quantifier-free diagram of $\mathcal{M}$ is $\Sigma^0_1$.
2. The closed $\Pi^N$ diagram of $\mathcal{M}$ is $\Pi^0_N$, and the open $\Pi^N$ diagram of $\mathcal{M}$ is $\Sigma^0_{N+1}$.
3. The closed $\Sigma^N$ diagram of $\mathcal{M}$ is $\Pi^0_{N+1}$, and the open $\Sigma^N$ diagram of $\mathcal{M}$ is $\Sigma^0_N$. 
Summary of Finitary Result(s)

Theorem (C., Goldbring, McNicholl)

There is a signature $L'$ and a computably presentable $L'$-structure $\mathcal{M}$ with the following properties:

1. The closed quantifier-free diagram of $\mathcal{M}$ is $\Pi^0_1$-complete, and the open quantifier-free diagram of $\mathcal{M}$ is $\Sigma^0_1$-complete.
2. For every positive integer $N$, the closed $\Pi^N$ diagram of $\mathcal{M}$ is $\Pi^0_N$-complete, and the open $\Pi^N$ diagram of $\mathcal{M}$ is $\Sigma^0_{N+1}$-complete.
3. For every positive integer $N$, the closed $\Sigma^N$ diagram of $\mathcal{M}$ is $\Pi^0_{N+1}$-complete, and the open $\Sigma^0_N$ diagram of $\mathcal{M}$ is $\Sigma^0_N$-complete.
Summary of Finitary Result(s)

Theorem (C., Goldbring, McNicholl)

There is a signature $L'$ and a computably presentable $L'$-structure $\mathcal{M}$ with the following properties:

1. The closed quantifier-free diagram of $\mathcal{M}$ is $\Pi^0_1$-complete, and the open quantifier-free diagram of $\mathcal{M}$ is $\Sigma^0_1$-complete.
Theorem (C., Goldbring, McNicholl)

There is a signature $L'$ and a computably presentable $L'$-structure $M$ with the following properties:

1. The closed quantifier-free diagram of $M$ is $\Pi^0_1$-complete, and the open quantifier-free diagram of $M$ is $\Sigma^0_1$-complete.

2. For every positive integer $N$, the closed $\Pi^0_N$ diagram of $M$ is $\Pi^0_{N+1}$-complete, and the open $\Pi^0_N$ diagram of $M$ is $\Sigma^0_{N+1}$-complete.
Summary of Finitary Result(s)

**Theorem (C., Goldbring, McNicholl)**

There is a signature $L'$ and a computably presentable $L'$-structure $M$ with the following properties:

1. The closed quantifier-free diagram of $M$ is $\Pi^0_1$-complete, and the open quantifier-free diagram of $M$ is $\Sigma^0_1$-complete.

2. For every positive integer $N$, the closed $\Pi^0_N$ diagram of $M$ is $\Pi^0_{N+1}$-complete, and the open $\Pi^0_N$ diagram of $M$ is $\Sigma^0_{N+1}$-complete.

3. For every positive integer $N$, the closed $\Sigma^0_N$ diagram of $M$ is $\Pi^0_{N+1}$-complete, and the open $\Sigma^0_N$ diagram of $M$ is $\Sigma^0_N$-complete.
Summary of Infinitary Result(s)

Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure and let $\varphi$ be a computable infinitary sentence of $L$.

1. If $\varphi$ is $\Pi^{c}_{\alpha}$, then $D > (\varphi_{\mathcal{M}})$ is $\Sigma^{0}_{\alpha}$ uniformly in a code of $\varphi$, and $D \geq (\varphi_{\mathcal{M}})$ is $\Pi^{0}_{\alpha + 1}$ uniformly in a code of $\varphi$.

2. If $\varphi$ is $\Sigma^{c}_{\alpha}$, then $D > (\varphi_{\mathcal{M}})$ is $\Sigma^{0}_{\alpha}$ uniformly in a code of $\varphi$, and $D \geq (\varphi_{\mathcal{M}})$ is $\Pi^{0}_{\alpha + 1}$ uniformly in a code of $\varphi$. 
Summary of Infinitary Result(s)

Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure and let $\varphi$ be a computable infinitary sentence of $L$.

1. If $\varphi$ is $\Pi^c_{\alpha}$, then $D^>(\varphi^\mathcal{M})$ is $\Sigma^0_{\alpha+1}$ uniformly in a code of $\varphi$, and $D^\geq(\varphi^\mathcal{M})$ is $\Pi^0_{\alpha}$ uniformly in a code of $\varphi$. 
Theorem (C., Goldbring, McNicholl)

Let $\mathcal{M}$ be a computably presentable $L$-structure and let $\varphi$ be a computable infinitary sentence of $L$.

1. If $\varphi$ is $\Pi^c_\alpha$, then $D^>(\varphi^\mathcal{M})$ is $\Sigma^0_{\alpha+1}$ uniformly in a code of $\varphi$, and $D^\geq(\varphi^\mathcal{M})$ is $\Pi^0_\alpha$ uniformly in a code of $\varphi$.

2. If $\varphi$ is $\Sigma^c_\alpha$, then $D^>(\varphi^\mathcal{M})$ is $\Sigma^0_\alpha$ uniformly in a code of $\varphi$, and $D^\geq(\varphi^\mathcal{M})$ is $\Pi^0_{\alpha+1}$ uniformly in a code of $\varphi$. 
Theorem (C., Goldbring, McNicholl)

There is a signature $L''$ and an $L''$-structure $\mathcal{M}$ so that the following hold for every computable ordinal $\alpha$.

1. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Pi^c_{\alpha}$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^{\geq}(\psi_i^\mathcal{M})\}$ is $\Pi^0_{\alpha}$-complete.

2. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Sigma^c_{\alpha}$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^{>}(\psi_i^\mathcal{M})\}$ is $\Sigma^0_{\alpha}$-complete.

3. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Pi^c_{\alpha}$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^{>}(\psi_i^\mathcal{M})\}$ is $\Sigma^0_{\alpha+1}$-complete.

4. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Sigma^c_{\alpha}$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^{\geq}(\psi_i^\mathcal{M})\}$ is $\Pi^0_{\alpha+1}$-complete.
Summary of Infinitary Result(s)

Theorem (C., Goldbring, McNicholl)

There is a signature $L''$ and an $L''$-structure $\mathcal{M}$ so that the following hold for every computable ordinal $\alpha$.

1. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Pi^c_{\alpha}$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^\geq(\psi_i^m)\}$ is $\Pi^0_{\alpha}$-complete.

2. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Sigma^c_{\alpha}$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^>(\psi_i^m)\}$ is $\Sigma^0_{\alpha}$-complete.
Summary of Infinitary Result(s)

**Theorem (C., Goldbring, McNicholl)**

There is a signature \( L'' \) and an \( L'' \)-structure \( \mathcal{M} \) so that the following hold for every computable ordinal \( \alpha \).

1. There is a computable sequence \( (\psi_i)_{i \in \mathbb{N}} \) of \( \Pi^c_\alpha \) sentences of \( L'' \) so that \( \{ i : \frac{1}{2} \in D \geq (\psi_i^m) \} \) is \( \Pi^0_\alpha \)-complete.

2. There is a computable sequence \( (\psi_i)_{i \in \mathbb{N}} \) of \( \Sigma^c_\alpha \) sentences of \( L'' \) so that \( \{ i : \frac{1}{2} \in D > (\psi_i^m) \} \) is \( \Sigma^0_\alpha \)-complete.

3. There is a computable sequence \( (\psi_i)_{i \in \mathbb{N}} \) of \( \Pi^c_\alpha \) sentences of \( L'' \) so that \( \{ i : \frac{1}{2} \in D > (\psi_i^m) \} \) is \( \Sigma^0_{\alpha+1} \)-complete.
Summary of Infinitary Result(s)

Theorem (C., Goldbring, McNicholl)

There is a signature $L''$ and an $L''$-structure $\mathcal{M}$ so that the following hold for every computable ordinal $\alpha$.

1. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Pi^c_\alpha$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^\geq(\psi_i^\mathcal{M})\}$ is $\Pi^0_\alpha$-complete.

2. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Sigma^c_\alpha$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^>(\psi_i^\mathcal{M})\}$ is $\Sigma^0_\alpha$-complete.

3. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Pi^c_\alpha$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^>(\psi_i^\mathcal{M})\}$ is $\Sigma^0_{\alpha+1}$-complete.

4. There is a computable sequence $(\psi_i)_{i \in \mathbb{N}}$ of $\Sigma^c_\alpha$ sentences of $L''$ so that $\{i : \frac{1}{2} \in D^\geq(\psi_i^\mathcal{M})\}$ is $\Pi^0_{\alpha+1}$-complete.
The results may be intuitive, but proving optimality was not!
The results may be intuitive, but proving optimality was not!

Two initial ideas:
The results may be intuitive, but proving optimality was not!

Two initial ideas:

1. True arithmetic with the discrete metric (\(=\)); realizes optimality in classical case.
The results may be intuitive, but proving optimality was not!

Two initial ideas:

1. True arithmetic with the discrete metric (=); realizes optimality in classical case.
2. \([0, 1]\) with the Euclidean metric; simplest continuous structure.
The results may be intuitive, but proving optimality was not!

Two initial ideas:

1. True arithmetic with the discrete metric ($=$); realizes optimality in classical case.

2. $[0, 1]$ with the Euclidean metric; simplest continuous structure.

But neither of these work!
Discussion of Diagram Result(s)

In the case of true arithmetic, since truth remains discretely valued (other than trivial application of the $\frac{1}{2}$ connective), we have the following.
Discussion of Diagram Result(s)

In the case of true arithmetic, since truth remains discretely valued (other than trivial application of the $\frac{1}{2}$ connective), we have the following.

1. Both the open and closed $\Pi_N$ diagrams are $\Pi^0_N$-complete.
Discussion of Diagram Result(s)

In the case of true arithmetic, since truth remains discretely valued (other than trivial application of the $\frac{1}{2}$ connective), we have the following.

1. Both the open and closed $\Pi_N$ diagrams are $\Pi^0_N$-complete.
2. Both the open and closed $\Sigma_N$ diagrams are $\Sigma^0_N$-complete.
In the case of $[0, 1]$ with the Euclidean metric, recall that the standard presentation of this structure is computably compact.

We proved the following.

Proposition (C., Goldbring, McNicholl)

Let $M^\#$ be a computably compact computable presentation of an $L$-structure $M$. Then the open diagram of $M$ is $\Sigma_0^1$ and the closed diagram of $M$ is $\Pi_0^1$.

This result follows from the standard result in computable analysis that maxima and minima of computable functions are computable on computably compact spaces. Thus we don't even achieve optimality in the simple cases!
Discussion of Diagram Result(s)

In the case of $[0, 1]$ with the Euclidean metric, recall that the standard presentation of this structure is computably compact. We proved the following.

Proposition (C., Goldbring, McNicholl)

Let $M$ be a computably compact computable presentation of an $L$-structure $M$. Then the open diagram of $M$ is $\Sigma^0_1$ and the closed diagram of $M$ is $\Pi^0_1$.

This result follows from the standard result in computable analysis that maxima and minima of computable functions are computable on computably compact spaces. Thus we don't even achieve optimality in the simple cases!
Discussion of Diagram Result(s)

In the case of $[0, 1]$ with the Euclidean metric, recall that the standard presentation of this structure is computably compact. We proved the following.

**Proposition (C., Goldbring, McNicholl)**

Let $\mathcal{M}^\#$ be a computably compact computable presentation of an $L$-structure $\mathcal{M}$. Then the open diagram of $\mathcal{M}$ is $\Sigma^0_1$ and the closed diagram of $\mathcal{M}$ is $\Pi^0_1$.

This result follows from the standard result in computable analysis that maxima and minima of computable functions are computable on computably compact spaces. Thus we don’t even achieve optimality in the simple cases!
Discussion of Diagram Result(s)

In the case of $[0, 1]$ with the Euclidean metric, recall that the standard presentation of this structure is computably compact. We proved the following.

**Proposition (C., Goldbring, McNicholl)**

Let $M^#$ be a computably compact computable presentation of an $L$-structure $M$. Then the open diagram of $M$ is $\Sigma^0_1$ and the closed diagram of $M$ is $\Pi^0_1$.

This result follows from the standard result in computable analysis that maxima and minima of computable functions are computable on computably compact spaces.
In the case of $[0, 1]$ with the Euclidean metric, recall that the standard presentation of this structure is computably compact. We proved the following.

**Proposition (C., Goldbring, McNicholl)**

Let $M^\#$ be a computably compact computable presentation of an $L$-structure $M$. Then the open diagram of $M$ is $\Sigma^0_1$ and the closed diagram of $M$ is $\Pi^0_1$.

This result follows from the standard result in computable analysis that maxima and minima of computable functions are computable on computably compact spaces. Thus we don’t even achieve optimality in the simple cases!
Discussion of Diagram Result(s)

Our space needed to be non-compact.
Our space needed to be non-compact. We then returned to the natural numbers with the discrete metric (in some sense, the simplest non-compact space).
Our space needed to be non-compact. We then returned to the natural numbers with the discrete metric (in some sense, the simplest non-compact space).

Since true arithmetic wouldn’t suffice, we constructed a new structure via a combinatorial lemma.
Discussion of Diagram Result(s)

**Theorem (C., Goldbring, McNicholl)**

Let \( R \subseteq \mathbb{N}^{N+2} \), and let \( n \in \mathbb{N} \).

1. \( n \in \forall R \) if and only if

\[
\inf_{x_1} \sup_{x_2} \ldots \sup_{x_N} \Gamma(1 - \frac{1}{2} \chi_{R^*}; x_1, \ldots, x_N, n) \leq \frac{1}{2}.
\]

2. \( n \in \exists R \) if and only if

\[
\sup_{x_1} \inf_{x_2} \ldots \inf_{x_N} \Gamma(\frac{1}{2} \chi_{(-R)^*}; x_1, \ldots, x_N, n) < \frac{1}{2}.
\]
Discussion of Diagram Result(s)

Theorem (C., Goldbring, McNicholl)

Let $R \subseteq \mathbb{N}^{N+2}$, and let $n \in \mathbb{N}$.

1. $n \in \forall R$ if and only if

$$\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} \Gamma\left(1 - \frac{1}{2} \chi_{R^*}; x_1, \ldots, x_N, n\right) \leq \frac{1}{2}.$$ 

2. $n \in \exists R$ if and only if

$$\sup_{x_1} \inf_{x_2} \ldots Q_{x_N} \Gamma\left(\frac{1}{2} \chi_{(-R)^*}; x_1, \ldots, x_N, n\right) < \frac{1}{2}.$$ 

$\forall R$ and $\exists R$ are just sets coded by relations $\Gamma$.
Theorem (C., Goldbring, McNicholl)

Let $R \subseteq \mathbb{N}^{N+2}$, and let $n \in \mathbb{N}$.

1. $n \in \overrightarrow{\forall} R$ if and only if

$$\inf_{x_1} \sup_{x_2} \ldots Q_{x_N} \Gamma(1 - \frac{1}{2} \chi_{R^*}; x_1, \ldots, x_N, n) \leq \frac{1}{2}.$$ 

2. $n \in \overrightarrow{\exists} R$ if and only if

$$\sup_{x_1} \inf_{x_2} \ldots Q_{x_N} \Gamma(\frac{1}{2} \chi_{(-R)^*}; x_1, \ldots, x_N, n) < \frac{1}{2}.$$ 

$\overrightarrow{\forall} R$ and $\overrightarrow{\exists} R$ are just sets coded by relations and $\Gamma$ is a special summation function.
Corollary

Let $\mathcal{M}$ be an $L$-structure with a computably compact computable presentation. Then the theory of $\mathcal{M}$ is $\Delta^0_2$. 
Corollary

Let $\mathcal{M}$ be an $L$-structure with a computably compact computable presentation. Then the theory of $\mathcal{M}$ is $\Delta^0_2$.

In Harrison-Trainor, Melnikov, and Meng Ng (2020), it was shown that any computable Stone space has a computably compact computable presentation. We thus attain the following.
Applications of Diagram Result(s)

Corollary

Let $\mathcal{M}$ be an $L$-structure with a computably compact computable presentation. Then the theory of $\mathcal{M}$ is $\Delta^0_2$.

In Harrison-Trainor, Melnikov, and Meng Ng (2020), it was shown that any computable Stone space has a computably compact computable presentation. We thus attain the following.

Corollary

Let $\mathcal{X}$ be a computable Stone space. Then the (continuous) theory of $\mathcal{X}$ is $\Delta^0_2$. 
Corollary

Let $\mathcal{M}$ be an $L$-structure with an (hyper)arithmetic presentation. Then the theory of $\mathcal{M}$ is also (hyper)arithmetic.
Corollary

Let $\mathcal{M}$ be an $L$-structure with an (hyper)arithmetic presentation. Then the theory of $\mathcal{M}$ is also (hyper)arithmetic.

This has already been applied in Goldbring and Hart (2020) in the proof of the following.
Applications of Diagram Result(s)

Theorem (Theorem 1.1, Goldbring and Hart, 2020)

The following operator algebras have hyperarithmetic theory.

1. The hyperfinite $\text{II}_1$ factor $\mathcal{R}$.
2. $L(\Gamma)$ for $\Gamma$ a finitely generated group with solvable word problem.
3. $C^*(\Gamma)$ for $\Gamma$ a finitely presented group.
4. $C^*_\lambda(\Gamma)$ for $\Gamma$ a finitely generated group with solvable word problem.
Future Work

- R.i.c.e. (relatively intrinsically computably enumerable) predicates (relations) in the continuous setting?
Future Work

- R.i.c.e. (relatively intrinsically computably enumerable) predicates (relations) in the continuous setting?
- Optimal bounds on all diagrams for the hyperfinite II$_1$ factor $\mathcal{R}$?
Future Work

- R.i.c.e. (relatively intrinsically computably enumerable) predicates (relations) in the continuous setting?
- Optimal bounds on all diagrams for the hyperfinite $\text{II}_1$ factor $\mathcal{R}$?
- Enforceable operator algebras and effective completeness?
In each of the following, $\mathcal{M}$ is a computably presentable metric structure. Whenever we refer to a set being open or closed, we mean with respect to the topology induced on the universe of $\mathcal{M}$ by its metric.
Future Work

In each of the following, $\mathcal{M}$ is a computably presentable metric structure. Whenever we refer to a set being open or closed, we mean with respect to the topology induced on the universe of $\mathcal{M}$ by its metric.

We currently investigate only unary predicates as a toy case (once this is proven, the results \textit{should} easily generalize).
Future Work

In each of the following, $\mathcal{M}$ is a computably presentable metric structure. Whenever we refer to a set being open or closed, we mean with respect to the topology induced on the universe of $\mathcal{M}$ by its metric.

We currently investigate only unary predicates as a toy case (once this is proven, the results *should* easily generalize).

**Definition**

Fix a computable presentation $\mathcal{M}^\#$ of $\mathcal{M}$. An open set $U \subseteq |\mathcal{M}|$ is a *c.e. open set* of $\mathcal{M}^\#$ if there is a computable sequence $(B_j)_{j \in \mathbb{N}}$ of rational open balls of $\mathcal{M}^\#$ such that $U = \bigcup_{j \in \mathbb{N}} B_j$. 
Future Work

**Definition**
An open set \( U \subseteq |M| \) is an *intrinsically c.e. open relation* if for every computable presentation \( M^\# \) of \( M \), \( U \) is a c.e. open set of \( M^\# \).
Future Work

Definition
An open set $U \subseteq |M|$ is an \textit{intrinsically c.e. open relation} if for every computable presentation $M\#$ of $M$, $U$ is a c.e. open set of $M\#$.

Definition
An open set $U \subseteq |M|$ is a \textit{relatively intrinsically c.e. open relation} (r.i.c.e. open) if there is some finite tuple $\bar{c} \in |M|^n$ such that for every computable presentation $(M, \bar{c})\#$ of $(M, \bar{c})$, there is an enumeration operator $\Phi$ such that for any enumeration $\gamma : \mathbb{N} \rightarrow D<(M, \bar{c})\#$, $\Phi(\gamma)$ is a sequence $(B_j)_{j \in \mathbb{N}}$ of rational open balls of $M\#$ such that $U = \bigcup_{j \in \mathbb{N}} B_j$. 
Future Work

**Definition**

An open set $U \subseteq |\mathcal{M}|$ is $\Sigma^c_1$-definable with parameters in $\mathcal{M}$ if there is some finite tuple $\bar{c} \in |\mathcal{M}|^n$ and a $\Sigma^c_1$-formula $\varphi$ such that

$$a \in U \iff (\varphi(a, \bar{c}))^\mathcal{M} < \frac{1}{2}.$$
Future Work

Definition
An open set $U \subseteq |\mathcal{M}|$ is $\Sigma^c_1$-definable with parameters in $\mathcal{M}$ if there is some finite tuple $\overline{c} \in |\mathcal{M}|^n$ and a $\Sigma^c_1$-formula $\varphi$ such that

$$a \in U \iff (\varphi(a, \overline{c}))^\mathcal{M} < \frac{1}{2}.$$ 

Conjecture
Let $U \subseteq |\mathcal{M}|$ be open. Then the following are equivalent.

(a) $U$ is a r.i.c.e. open relation.

(b) $U$ is $\Sigma^c_1$-definable with parameters in $\mathcal{M}$.
QUESTIONS?
Works Cited


Works Cited


Works Cited