Computable Structure Theory of Continuous Logic

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3. Generalized effective completeness.
4. Computable infinitary continuous logic and hyperarithmetic numerals.
5. Diagram complexity of a computably presented metric structure.
6. Future work on r.i.c.e. relations in the continuous setting.

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If $d(x, y) < d(x, z)$, then y is *closer* to x than z is to x.

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Since $d(x,y)=0$ if and only if $x=y$, "0" corresponds to truth, while "1" to falsity.

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- $P(t_0, \ldots, t_{\eta(P)-1})$
- $\neg\varphi$, $\frac{1}{2}\varphi$, and $\varphi \dot{-} \psi$, where φ and ψ are also quantifier-free.

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- $\sup_{x_0} \varphi$ is Π_1.
- $\inf_{x_1} \sup_{x_0} \varphi$ is Σ_2.
- $\sup_{x_{N-1}} \inf_{x_{N-2}} \ldots \varphi$ is Π_N.

Continuous Logic

Shorthand	**String**
$\varphi \vee \psi$	$\neg((\neg\varphi) \dot{-} \psi)$
$\varphi \wedge \psi$	$\varphi \dot{-} (\varphi \dot{-} \psi)$
$\varphi \leftrightarrow \psi$	$(\varphi \dot{-} \psi) \vee (\psi \dot{-} \varphi)$
$\underline{0}$	$\sup_x \underline{d}(x, x)$
$\underline{1}$	$\neg\, \underline{0}$
$\varphi \dot{+} \psi$	$\neg((\underline{1} \dot{-} \varphi) \dot{-} \psi))$
$m\varphi$	$\underbrace{(...(\varphi \dot{+} \varphi) \dot{+} \cdots \dot{+} \varphi)}_{m\text{-many}}$
$\underline{2^{-k}}$	$\underbrace{\frac{1}{2} \cdots \frac{1}{2}}_{k\text{-many}} \underline{1}$

Infinitary continuous logic

For a countable index set I, if $(\varphi_i)_{i \in I}$ share a tuple of free variables and are uniformly equicontinuous in those variables, then

$$\bigwedge_{i \in I} \varphi_i \quad \text{and} \quad \bigvee_{i \in I} \varphi_i$$

are infinitary formulas.

Metric structures

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$$\mathfrak{M} = \left(|\mathfrak{M}|, d, \{P^{\mathfrak{M}} : P \in \mathcal{P}\}, \{f^{\mathfrak{M}} : f \in \mathcal{F}\}, \{c^{\mathfrak{M}} : c \in \mathcal{C}\}\right),$$

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If, moreover, $(|\mathfrak{M}|, d)$ is a complete metric space, then $\mathfrak{M}$ is an *L-structure*.

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When Γ is a set of wffs, $\mathfrak{M} \models \Gamma$ means $\mathfrak{M} \models \varphi$ for every $\varphi \in \Gamma$.

Metric structures

Example

The unit ball of a Banach space over $\mathbb{R}$, the metric induced by the norm, as functions all binary maps of the form

$$f_{\alpha,\beta}(x, y) = \alpha x + \beta y$$

where $|\alpha| + |\beta| \leq 1$ as scalars, and the additive identity 0 and some choice of normal basis $(e_i)_{i\in I}$ as distinguished points.

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Metric structures

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In this case, the metric just serves to indicate equality.

Presentations

Given a structure $\mathfrak{M}$ and $A \subseteq |\mathfrak{M}|$, we define the *algebra generated by* A to be the smallest subset of $|\mathfrak{M}|$ containing A that is closed under every function of $\mathfrak{M}$.

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A pair $(\mathfrak{M}, g)$ is called a *presentation* of $\mathfrak{M}$ if $g : \mathbb{N} \to |\mathfrak{M}|$ is a map such that the algebra generated by $\mathrm{ran}(g)$ is dense.

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Each point in the algebra generated by the distinguished points is called a *rational point* of $\mathfrak{M}^\sharp$.

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- Δ_n^0 if it is both Σ_n^0 and Π_n^0.

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A set $A \subseteq \mathbb{N}$ is *hyperarithmetical* if it is Σ^0_α for some computable ordinal α.

Computable presentations

Definition

Let A be a countable set. A map $f : A \to \mathbb{R}$ is *computable* if there is an effective procedure which, given $a \in A$ and $k \in \mathbb{N}$, outputs a rational q such that

$$|f(a) - q| < 2^{-k}.$$

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We say that a metric structure is *computably presentable* if it has a computable presentation.

Motivation for First Result

Foundations of recursive model theory, Terrence Millar, 1978.

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Theorem (Effective Completeness)

In classical logic, if a theory is decidable (meaning its set of consequences is computable), then it is modeled by a computable structure.

Generalized Effective Completeness

Definition (Ben Yaacov and Pedersen, 2010)

Let Γ be a set of wffs. The *degree of truth with respect to* Γ ($\cdot^{\circ}_{\Gamma}$) is a map from wffs to $[0, 1]$, defined as

$$\varphi^{\circ}_{\Gamma} := \sup \left\{ \varphi^{\mathfrak{M}} : \mathfrak{M} \models \Gamma \right\}.$$

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Definition (Ben Yaacov and Pedersen, 2010)

A theory T is *decidable* if $\cdot^{\circ}_{T}$ is a computable map from $(\varphi_n)_{n\in\mathbb{N}}$ into $[0,1]$.

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- For every $n, k, m \in \mathbb{N}$, if $\langle n, k, m \rangle \in \mathrm{ran}(X)$, then $q_m \in \left[\ (\varphi_n)^\circ_T - 2^{-k}\ ,\ (\varphi_n)^\circ_T + 2^{-k}\ \right]$.

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Proposition

A theory is decidable if and only if it has a computable name.

Generalized Effective Completeness

Lemma

There is an effective procedure which given X, a name of an L-theory T, outputs $\Phi(X) \subseteq \mathbb{N}$ such that $T \cup \{\theta_n : n \in \Phi(X)\}$ is consistent, and for every pair of wffs φ and ψ, either φ is provably equivalent to ψ, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\{\theta_n : n \in \Phi(X)\}$.

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There is an effective procedure which, given X, a name of an L-theory T, produces a presentation of a structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.

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Corollary (Effective Completeness of Continuous Logic)

Every decidable theory is modeled by a computably presentable structure.

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- $\underline{0} := \sup_x d(x,x)$.
- $\underline{1} := \neg \underline{0}$.
- $\underline{\frac{1}{2^k}} := \frac{1}{2} \dots \frac{1}{2}\ \underline{1}$. ($k$-many $\frac{1}{2}$ connectives)
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Motivation for Second Result

Ben Yaacov and Pedersen (2010) noted that for any dyadic $r \in [0,1]$ (*i.e.*, number of the form $\frac{\ell}{2^k}$), there is a finitary sentence which is universally interpreted as r. (Recall the shorthands given previously.)

- $\underline{0} := \sup_x d(x,x)$.
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But what if we consider computable infinitary formulas?

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- The $\Sigma_0^c = \Pi_0^c$ sets include all quantifier-free, finitary wffs.
- A wff φ is Σ_α^c if it is of the form

$$\varphi = \bigwedge_{i \in I} \inf_{\vec{x}} \psi_i$$

where $I \subseteq \mathbb{N}$ is c.e., each $\psi_i \in \Pi_\beta^c$, for some $\beta < \alpha$, and a modulus of continuity for φ exists that is computable from a code for φ.

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Hyperarithmetical Numerals

Theorem (C., McNicholl)

There is an effective procedure which, given a hyperarithmetical right Dedekind cut of a real number $r \in [0, 1]$, *outputs a computable infinitary sentence such that the following hold.*

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- *If the right Dedekind cut given is Π^0_α, then the output is a Π^c_α sentence φ such that for every structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$.*
- *If the right Dedekind cut given is Σ^0_α, then the output is a Σ^c_α sentence φ such that for every structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$.*

Discussion of Second Result

In other words, for any nonzero computable ordinal α and any right Π^0_α (or Σ^0_α) real number $r \in [0, 1]$, there is a Π^c_α (respectively, Σ^c_α) sentence φ such that for *every* metric structure $\mathfrak{M}$, $\varphi^{\mathfrak{M}} = r$.

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Corollary (C., McNicholl)

Suppose $\mathfrak{M}$ is an interpretation of $L^c_{\omega_1\omega}$, and suppose X computes the continuous theory of $\mathfrak{M}$. Then, X computes every hyperarithmetic set. Thus, no hyperarithmetic set computes the continuous theory of any metric structure.

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Definition

Let $(r_n)_{n\in\omega}$ be a sequence of real numbers and $g : \omega \to \omega_1^{\mathrm{CK}}$. We say $(r_n)_{n\in\omega}$ is **weakly uniformly right** Σ^0_g (Π^0_g) if there is a computable function $f : \omega \to \omega$ such that for all $n \in \omega$, $f(n)$ is a $\Sigma^0_{g(n)}$ ($\Pi^0_{g(n)}$) index of a right Dedekind cut of r_n.

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3. *If α is a limit ordinal and a right Dedekind cut of s is Σ^0_α, then there is a computable map $h : \omega \to \alpha$ and a sequence of real numbers $(r_n)_{n\in\omega}$ such that $s = \inf_{n\in\omega} r_n$ and $(r_n)_{n\in\omega}$ is weakly uniformly right Π^0_h.*

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Similarly, we can also ask the same question about *computable infinitary* sentences in continuous logic.

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E.g. The classical Σ_1 diagram of a structure describes the truth or falsity of every Σ_1 sentence.

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The Π_N diagrams relativize.

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The problem of uniformly deciding if one computable real number is less than *or equal* to another is Π^0_1-complete.

Why? To check equality, you would need to computably check *every* digit of each number to guarantee they all match.

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2. $[0, 1]$ with the Euclidean metric; simplest continuous structure.

Discussion of Diagram Result(s)

The results may be intuitive, but proving optimality was not!

Two initial ideas:

1. True arithmetic with the discrete metric $(=)$; realizes optimality in classical case.
2. $[0, 1]$ with the Euclidean metric; simplest continuous structure.

But neither of these work!

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Proposition (C., Goldbring, McNicholl)

Let $\mathfrak{M}^\sharp$ be a computably compact computable presentation of an L-structure $\mathfrak{M}$. Then the open diagram of $\mathfrak{M}$ is Σ^0_1 and the closed diagram of $\mathfrak{M}$ is Π^0_1.

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This result follows from the standard result in computable analysis that maxima and minima of computable functions are computable on computably compact spaces. Thus we don't even achieve optimality in the simple cases!

Discussion of Diagram Result(s)

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Since true arithmetic wouldn't suffice, we constructed a new structure via a combinatorial lemma.

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Theorem (C., Goldbring, McNicholl)

Let $R \subseteq \mathbb{N}^{N+2}$, and let $n \in \mathbb{N}$.

1. $n \in \vec{\forall} R$ *if and only if*

$$\inf_{x_1} \sup_{x_2} \dots Q_{x_N} \Gamma(1 - \frac{1}{2}\chi_{R^*}; x_1, \dots, x_N, n) \leq \frac{1}{2}.$$

2. $n \in \vec{\exists} R$ *if and only if*

$$\sup_{x_1} \inf_{x_2} \dots Q_{x_N} \Gamma(\frac{1}{2}\chi_{(\neg R)^*}; x_1, \dots, x_N, n) < \frac{1}{2}.$$

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$\vec{\forall} R$ and $\vec{\exists} R$ are just sets coded by relations and Γ is a special summation function.

Applications of Diagram Result(s)

Corollary

Let $\mathfrak{M}$ be an L-structure with a computably compact computable presentation. Then the theory of $\mathfrak{M}$ is Δ_2^0.

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In Harrison-Trainor, Melnikov, and Meng Ng (2020), it was shown that any computable Stone space has a computably compact computable presentation. We thus attain the following.

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In Harrison-Trainor, Melnikov, and Meng Ng (2020), it was shown that any computable Stone space has a computably compact computable presentation. We thus attain the following.

Corollary

Let $\mathcal{X}$ be a computable Stone space. Then the (continuous) theory of $\mathcal{X}$ is Δ^0_2.

Applications of Diagram Result(s)

Corollary

Let $\mathfrak{M}$ be an L-structure with an (hyper)arithmetic presentation. Then the theory of $\mathfrak{M}$ is also (hyper)arithmetic.

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This has already been applied in Goldbring and Hart (2020) in the proof of the following.

Applications of Diagram Result(s)

Theorem (Theorem 1.1, Goldbring and Hart, 2020)

The following operator algebras have hyperarithmetic theory.
(1) The hyperfinite II_1 factor $\mathcal{R}$.
(2) $L(\Gamma)$ for Γ a finitely generated group with solvable word problem.
(3) $C^*(\Gamma)$ for Γ a finitely presented group.
(4) $C^*_\lambda(\Gamma)$ for Γ a finitely generated group with solvable word problem.

Future Work

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- Optimal bounds on all diagrams for the hyperfinite II_1 factor $\mathcal{R}$?
- Enforceable operator algebras and effective completeness?

Future Work

In each of the following, $\mathfrak{M}$ is a computably presentable metric structure. Whenever we refer to a set being open or closed, we mean with respect to the topology induced on the universe of $\mathfrak{M}$ by its metric.

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Definition

Fix a computable presentation $\mathfrak{M}^\sharp$ of $\mathfrak{M}$. An open set $U \subseteq |\mathfrak{M}|$ is a *c.e. open set* of $\mathfrak{M}^\sharp$ if there is a computable sequence $(B_j)_{j\in\mathbb{N}}$ of rational open balls of $\mathfrak{M}^\sharp$ such that $U = \bigcup_{j\in\mathbb{N}} B_j$.

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Definition

An open set $U \subseteq |\mathfrak{M}|$ is an *intrinsically c.e. open relation* if for every computable presentation $\mathfrak{M}^\sharp$ of $\mathfrak{M}$, U is a c.e. open set of $\mathfrak{M}^\sharp$.

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Definition

An open set $U \subseteq |\mathfrak{M}|$ is a *relatively intrinsically c.e. open relation* (r.i.c.e. open) if there is some finite tuple $\overline{c} \in |\mathfrak{M}|^n$ such that for every computable presentation $(\mathfrak{M}, \overline{c})^\sharp$ of $(\mathfrak{M}, \overline{c})$, there is an enumeration operator Φ such that for any enumeration $\gamma : \mathbb{N} \to D^<((\mathfrak{M}, \overline{c})^\sharp)$, $\Phi(\gamma)$ is a sequence $(B_j)_{j\in\mathbb{N}}$ of rational open balls of $\mathfrak{M}^\sharp$ such that $U = \bigcup_{j\in\mathbb{N}} B_j$.

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Definition

An open set $U \subseteq |\mathfrak{M}|$ is *Σ_1^c-definable with parameters in* $\mathfrak{M}$ if there is some finite tuple $\overline{c} \in |\mathfrak{M}|^n$ and a Σ_1^c-formula φ such that

$$a \in U \iff (\varphi(a, \overline{c}))^{\mathfrak{M}} < \frac{1}{2}.$$

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An open set $U \subseteq |\mathfrak{M}|$ is Σ_1^c*-definable with parameters in* $\mathfrak{M}$ if there is some finite tuple $\overline{c} \in |\mathfrak{M}|^n$ and a Σ_1^c-formula φ such that

$$a \in U \iff (\varphi(a, \overline{c}))^{\mathfrak{M}} < \frac{1}{2}.$$

Conjecture

Let $U \subseteq |\mathfrak{M}|$ be open. Then the following are equivalent.

(a) U is a r.i.c.e. open relation.

(b) U is Σ_1^c-definable with parameters in $\mathfrak{M}$.

QUESTIONS?

Works Cited

- Gödel, K. (1929). "Uber die Vollstandigkeit des Logikkalkuls". Doctoral dissertation. University Of Vienna.
- Henkin, L. (1947). "The completeness of formal systems". Doctoral dissertation. Princeton University.
- Millar, T. (1978). "Foundations of recursive model theory". *Annals of Mathematical Logic*, 13:45–72.
- Ben Yaacov, I. and Pedersen, A. P. (2010). "A proof of completeness for continuous first-order logic". The Journal of Symbolic Logic, 75(1):168–190.

Works Cited

- Ben Yaacov, I., Berenstein, A., Henson, C. W., and Usvyatsov, A. (2008). *Model theory for metric structures*. In *Model theory with applications to algebra and analysis*. Vol. 2, volume 350 of London Math. Soc. Lecture Note Ser., pages 315–427. Cambridge Univ. Press, Cambridge.
- Calvert, W. (2011). "Metric structures and probabilistic computation". *Theor. Comput. Sci.*, 412(25):2766–2775.
- Didehvar, F., Ghasemloo, K., and Pourmahdian, M. (2010). "Effectiveness in rpl, with applications to continuous logic". *Annals of Pure and Applied Logic*, 161(6):789–799.

Works Cited

- Harrison-Trainor, M., Melnikov, A., and Meng Ng, K. (2020). "Computability of polish spaces up to homeomorphism". *The Journal of Symbolic Logic*, 85(4):1664–1686.
- Goldbring, I. and Hart, B. (2020). "Operator algebras with hyperarithmetic theory". *Journal of Logic and Computation*, 31(2):612–629.
- C., Goldbring, I., and McNicholl, T. (2023). "On the complexity of the theory of a computably presented metric structure". *Archive for Mathematical Logic*, 62 (7):1111–1129.
- C. (2023). "Generalized effective completeness for continuous logic". *Journal of Logic and Analysis*, 15:4, 1–17.
- C. and McNicholl, T. (2023+) "Hyperarithmetic numerals". *preprint on arXiv.*