Probabilistic R-Macs

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A family $(M_k : k \in \omega)$ of finite L-structures is a *one-dimensional asymptotic class* if for every L-formula $\varphi(x_1, \ldots, x_n, \bar{y})$, there are L-formulas $\pi_1(\bar{y}), \ldots, \pi_r(\bar{y})$, pairs $(\mu_1, d_1), \ldots, (\mu_r, d_r) \in \mathbb{R} \times \{1, \ldots, n\}$ and a number $C \in \mathbb{R}$ such that

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Theorem (Chatzidakis, Van den Dries & Macintrye 92)

The class of finite fields is a one-dimensional asymptotic class.

Definition (Anscombe, Macpherson, Steinhorn, Wolf 16)

Let R be a set of functions $h : \{M : M \text{ is an } L\text{-structure}\} \to \mathbb{R}$. A sequence $(M_k : k \in \omega)$ of L-structures is an *R-multidimensional asymptotic class (R-mac)* if for every L-formula $\varphi(\bar{x}, \bar{y})$, there are L-formulas $\pi_1(\bar{y}), \ldots, \pi_r(\bar{y})$ and functions $h_1, \ldots, h_r \in R$ such that

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An N-dimensional asymptotic class (introduced by Elwes) is an R-mac where R is the set of functions $\{M \mapsto \mu|M|^{d/N} : \mu \in \mathbb{R}, d \in \mathbb{N}\}$.

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In this talk, all random structures considered will be finite – so there is no care we need to take with measurability issues, and we may consider the probability of any property of the random structure $\hat{M}$.

Definition

Let $n \in \omega$ and $p \in [0, 1]$. The Erdős-Rényi random graph $\hat{G}(n, p)$ is the random graph on n vertices formed by letting each of the $\binom{n}{2}$ possible edges appear independently with probability p.

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Definition

Let $\alpha \in (0,1)$. The *Spencer-Shelah random graph sequence with parameter* α is the sequence $(\hat{G}_n^\alpha := \hat{G}(n, n^{-\alpha}) : n \in \omega)$.

Definition

Let $(M_n : n \in \omega)$ be a sequence of random L-structures.
The *zero-one theory* of the sequence is the set of L-sentences

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Theorem (Spencer & Shelah)

If α is irrational, the Spencer-Shelah random graph sequence with parameter α has a complete zero-one theory T_α.

Definition (V.)

Let R be a set of functions from $\{M : M \text{ is an } L\text{-structure}\}$ to $\mathbb{R}$. Let $\varphi(\bar{x}; \bar{y})$ be an L-formula.

A sequence $(\hat{M}_n : n \in \omega)$ of random L-structures is a *probabilistic R-mac for φ* if there are L-formulas $\pi_1(\bar{y}), \ldots, \pi_r(\bar{y})$ and functions $h_1, \ldots, h_r \in R$ such that for every $\epsilon > 0$, the probabilities of the following statements go to 1 as n goes to infinity:

- the sets $\pi_1(M_n^{|\bar{y}|}), \ldots, \pi_r(\hat{M}_n^{|\bar{y}|})$ partition $\hat{M}_n^{|\bar{y}|}$, and
- if $\bar{b} \in \pi_i(\hat{M}_n^{|\bar{y}|})$ then $(1-\epsilon)h_i(\hat{M}_n) < |\varphi(\hat{M}_n^{|\bar{x}|}, \bar{b})| < (1+\epsilon)h_i(\hat{M}_n)$

$(\hat{M}_n : n \in \omega)$ is a *probabilistic R-mac* if it is a probabilistic R-mac for every formula $\varphi(\bar{x}, \bar{y})$.

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Note: The functions h_i are defined on (deterministic) L-structures; the expression $h_i(\hat{M}_n)$ is a real-valued random variable.

Proposition

Let $(\hat{M}_n : n \in \omega)$ be a probabilistic R-mac. For each n, let $X_{n,1}, X_{n,2}, \ldots$ be a sequence of statements (events) about the structure $\hat{M}_n$. Suppose that for each k, we have

$$\liminf_{n\to\infty} \mathbb{P}(\hat{M}_n \text{ satisfies } X_{n,1}, X_{n,2}, \ldots, \text{ and } X_{n,k}) > 0.$$

Then we can find an R-mac $(M_1, M_2, \ldots)$ such that for every k, M_n satisfies $X_{n,k}$ for cofinitely many n.

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Note

For example, we can ensure that the zero-one theory of the $\hat{M}_n$ is (contained in) the limit theory of the M_n.

Proposition

To show that a sequence $(\hat{M}_n : n \in \omega)$ *of random structures is a probabilistic* R*-mac, it suffices to show that it is a probabilistic* R*-mac for every formula* $\varphi(x_1; \bar{y})$ *with a single object variable (provided that* R *is asymptotically closed under addition and multiplication).*

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Proof.

Routine fiber-decomposition proof. To show R is a probabilistic R-mac for $\varphi(x_1 \dots x_k; \bar{y})$, re-contextualize the variables as $\varphi(x_1; x_2 \dots x_k \bar{y})$ and obtain estimates for $|\varphi(\hat{M}_n; a_2 \dots a_k \bar{b})|$, definably for $a_2 \dots a_k \bar{b})$ by formulas $\pi_i(x_2 \dots x_k \bar{y})$. By induction, obtain cardinality bounds for $|\pi_i(\hat{M}_n^{k-1}, \bar{b})|$, and use these to bound the size of $|\varphi(\hat{M}_n^k, \bar{b})|$. □

If $\mathcal{C}$ is a class of (deterministic) finite structures, we say $\mathcal{C}$ is a probabilistic R-mac if $(\hat{M}_n : n \in \omega)$ is, where $\hat{M}_n$ is the uniform distribution on all structures in $\mathcal{C}$ with universe $\{1, \ldots, n\}$.

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Proposition

The class of all finite graphs is a probabilistic R-mac, where R is the set $\{M \mapsto \mu|M|^d : \mu \in \mathbb{R}, d \in \mathbb{N}\}$ (a probabilistic one-dimensional asymptotic class).

Proof Sketch

- The sequence of uniform distributions for finite graphs is the same as the Erdős-Rényi sequence $(\hat{G}(n, 1/2) : n \in \omega)$.

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- By quantifier elimination and the one-variable lemma, it suffices to check the R-pmac condition for formulas of the form $\phi(x, \bar{y}) = \bigwedge_i xEy_i \wedge \bigwedge_j \neg xEy_j \wedge \rho(x, \bar{y})$, where $\rho(x, \bar{y})$ expresses that all elements are distinct.

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- That is, we wish to obtain estimates for $|\{a \in \hat{G}_n : \hat{G}_n \models \phi(a, \bar{b})\}|$ for $\bar{b} \in \hat{G}_n$.
- Assuming that all b_i are distinct, the probability that a given $a \notin \bar{b}$ satisfies $\phi(a, \bar{b})$ is $2^{-|\bar{y}|}$. Furthermore, these events are mutually independent for different a.

Hoeffding's Inequality (Bernoulli Version)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables. Let $S_n = \sum_{i \le n} X_i$. Then $\mathbb{P}(|S_n - \mathbb{E}[S_n]| \ge t) \le e^{-2t^2/n}$.

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- For a given ϵ, the probability that this number is *not* between $(1-\epsilon)2^{-k}n$ and $(1+\epsilon)2^{-k}n$ is $\le exp(-2(\epsilon 2^{-k} n)^2/n) = exp(-\eta n)$ for $\eta = 2^{-2k-1}\epsilon$. This goes to 0 as $n \to \infty$.

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- The uniform distribution on *these* partial orders can be modeled as an independent-coin-flip random structure, similar to the random graph.
- Therefore single-variable definable sets in the random partial order on $\{1, \ldots, n\}$ have cardinality approximately $2^{-k}n$ for some k, by the same argument as in the random graph.

Theorem

For irrational $\alpha \in (0,1)$*, the Spencer-Shelah random graph sequence is a probabilistic* R^α*-mac, where* R^α *is the set of functions* $\{M \mapsto k|M|^{a-b\alpha} : k, a, b \in \mathbb{N}\}$.

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- Notation: For a graph B on $\{1, \ldots, k\}$, let $\Delta_B(v_1 \ldots v_k)$ be the formula $\bigwedge_{iEj} v_i E v_j \wedge \bigwedge_{\neg iEj} \neg v_i E v_j$. Let $\Delta_B^+(v_1 \ldots v_k)$ be $\bigwedge_{iEj} v_i E v_j$.

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Proof Sketch Continued

- Observe: complete quantifier-free formulas are formulas of the above form for $|\bar{z}| = 0$.

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- By quantifier elimination, it suffices to show that $(\hat{G}_n^\alpha : n \in \omega)$ is an R^α-pmac for conjunctions of formulas of the form $\phi(x, \bar{y}) := \exists \bar{z} \Delta_B(x, \bar{y}, \bar{z})$ and their negations.

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- We first show asymptoticity results for quantifier-free formulas (i.e. the case $|\bar{z}| = 0$)

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- We first show asymptoticity results for quantifier-free formulas (i.e. the case $|\bar{z}| = 0$)
- Given $\bar{b} \in \hat{G}_n^\alpha$, we show how to write the cardinality of $\{a : \bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z})\}$ as a linear combination (possibly with negative coefficients) of cardinalities $|\{\bar{c} : \Delta_C(\bar{b}, \bar{c})\}|$ as C ranges over finitely many graphs.

Theorem

For irrational $\alpha \in (0,1)$, the Spencer-Shelah random graph sequence is a probabilistic R^α-mac, where R^α is the set of functions $\{M \mapsto k|M|^{a-b\alpha} : k, a, b \in \mathbb{N}\}$.

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- We then sketch how to deal with including conjuncts of the form $\neg\exists \bar{z} \Delta_B(x, \bar{b}, \bar{z})$.

Definition

Let $A \subseteq B$ be graphs. $\delta_\alpha(B/A)$ is the quantity $v - e\alpha$, where v is the number of vertices in $B \setminus A$ and e is the number of edges in B which do not have both endpoints in A.

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Let $A \subseteq B$ be graphs.
We say that B is *safe over* A if for every $A \subset C \subseteq B \subseteq$, $\delta(C/A) > 0$.
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- If $\delta(A/\emptyset) < 0$, then G contains no copy of A
- If $A \subseteq B$ is a safe extension, then every copy of A extends to a copy of B.
- If $A \subseteq B$ is a rigid extension, then every copy of A extends to at most $K(B/A)$ copies of B (for some fixed number $K(B/A)$).

Proposition (Spencer & Shelah)

Let B be a graph on $\{1, \ldots, k, k+1, \ldots, k+l\}$, and let A be the subgraph on $\{k+1, \ldots, k+l\}$. Then for any $\bar{b} \in \Delta^+(G_n^l)$ ($\bar{b} \in \Delta(G_n^l)$), $\mathbb{E}[|\Delta_B^+(G_n^k, \bar{b})|] \approx n^{\delta(B/A)}$ ($\approx \mathbb{E}[|\Delta_B(G_n^k, \bar{b})|]$).

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Proof.

There are $(n-l)(n-l-1)\ldots(n-l-(k-1)) \approx n^k$ extensions of $\bar{b}$ to $(k+l)$-tuples $\bar{a}\bar{b}$. For any such $(k+l)$-tuple to be an element of $\Delta_B^+(G_n^k, \bar{b})$, we need each of the e edges of B to occur, where e is the number in $\delta(B/A) = v - e\alpha$. The probability that this happens is $(n^{-\alpha})^e = n^{-e\alpha}$. By linearity of expectation, the expected number of elements in $\Delta^+(G_n^l)$ is $\approx n^k \cdot n^{-e\alpha} = n^{k-e\alpha} = n^{\delta(B/A)}$.
For $\delta_B(G_n^k, \bar{b})$, we note that $\bar{a} \in \delta_B(G_n^k, \bar{b})$ iff $\bar{a} \in \Delta_B^+(G_n^k, \bar{b})$ and $\bar{a} \notin \Delta_{B'}^+(G_n^k, \bar{b})$ for any graph B' for which B is a proper spanning subgraph, and for any such B', $\delta(B'/A) \leq \delta(B/A) - \alpha$. □

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However, this is *different* from saying that the actual number of extensions of $\bar{b}$ is asymptotically equal to $n^{\delta(B/A)}$. This is not generally the case: if B is not safe over A, then *most* copies of A do not extend to B, even though the expected number of copies is positive.

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On the other hand...

Theorem (Kim & Vu (2000))

Let B be a graph on $\{1, \ldots, k, k+1, \ldots, k+l\}$ and let A be the subgraph on $\{k+1, \ldots, k+l\}$. Suppose that B is safe over A. Then there is a positive constant $\epsilon > 0$ such that the probability of the statement

$$\text{for all } \bar{b} \in \Delta_A^+(G_n^l), n^{\delta(B/A)} \cdot (1 - n^{-\epsilon}) < |\Delta_B^+(G_n^k, \bar{b})| < n^{\delta(B/A)} \cdot (1 + n^{-\epsilon})$$

goes to 1 as n goes to ∞.

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$(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic R^α-mac for the formula $\Delta_B^+(\bar{x}; \bar{y})$, with measuring functions $G \mapsto 0$ and $h : G \mapsto |G|^{B/A}$.

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Proof.

For every ϵ, with probability approaching 1: if $\hat{G}_n \models \Delta_A^+(\bar{b})$ then $(1-\epsilon)h(\hat{G}_n^\alpha) < |\Delta_B((\hat{G}_n^\alpha)^{|\bar{x}|}, \bar{b})| < (1+\epsilon)h(\hat{G}_n^\alpha)$. If $\hat{G}_n \not\models \Delta_A^+(\bar{b})$ then $|\Delta_B((\hat{G}_n^\alpha)^{|\bar{x}|}, \bar{b})| = 0$. □

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Define $\delta^\star(B/A)$ to be $\delta(B/R)$, where $R = rs(A, B)$. Define $K^\star(B/A)$ to be $K(R/A)$ (the maximum number of extensions of a copy of A to a copy of R in T^α)

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Let B be a graph. Then $(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic R^α-mac for $\Delta_B^+(\bar{x},\bar{y})$.

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Let B be a graph on $\{1, \ldots, k+l\}$. Let A be the induced subgraph on $\{k+1, \ldots, k+l\}$. Then $(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic R^α-mac for $\Delta_B^+(x_1 \ldots x_k; y_1 \ldots y_l)$, with measuring functions $G \mapsto m|G|^{\delta^\star(B/A)}$ with $m \in \{0, 1, \ldots, K^\star(B/A)\}$.

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- If $\bar{b}$ has m extensions to R, then each of those extensions has approximately $n^{\delta(B/R)}$ extensions to B, so $\bar{b}$ has in total approximately $mn^{\delta(B/R)}$ extensions to B. That is, $|\Delta_A^+(\hat{G}_n^{|\bar{x}|}, \bar{b})|$ is approximately $mn^{\delta(B/R)}$.

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- Sketch of problem and solution: if we only add edges to the rigid part R of B over A, it is possible to have $\delta(B'/R) = \delta(B/R)$. In this case, every extension of R' (the rigid part of B') to B' is already an extension of R to B (since we are adding no new edges from R to the safe part of B). Therefore in the inclusion-exclusion formula, when we subtract this number of extensions, we are cancelling out an earlier term in its entirety, not asymptotically as in the earlier bullet point. (Board example)

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- For a given $\bar{b}$, we wish to count the number of a such that $\hat{G}_n^\alpha \models \bigvee_j \exists \bar{z} \Delta_{A_j}(a, \bar{b}, \bar{z})$.
- To do this, we enumerate every instance of a occuring in a copy of some A_j over $\bar{b}$.

- We now show how to find the cardinality of $\{a : \bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z})\}$.
- Note: by the theory T^α, it suffices to consider the case where each B_i is rigid over the graph on $a, \bar{b}$.
- Given $B_1, \ldots, B_k$ as above, we can find $A_1, \ldots, A_l$ such that for any $a, \bar{b}$, $\bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z})$ if and only if $\bigvee_j \exists \bar{z} \Delta_{A_j}(a, \bar{b}, \bar{z})$. (Picture proof on board)
- For a given $\bar{b}$, we wish to count the number of a such that $\hat{G}_n^\alpha \models \bigvee_j \exists \bar{z} \Delta_{A_j}(a, \bar{b}, \bar{z})$.
- To do this, we enumerate every instance of a occuring in a copy of some A_j over $\bar{b}$.
- By inclusion-exclusion principle, we add up all instances of a which occur at least once, subtract the number of instances of a occuring twice, add the number of instances of a occuring three times...

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- By rigidity, there is some K such that no $a\bar{b}$ extends to a copy of C where C can be written as a union of $\geq K$ instances of A_i's – since the number of copies of $a\bar{b}$ to any one A_i is (with probability -¿) bounded by some K_i. So this process terminates!

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- We are left with

$$\{a : \bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z})\}| = \sum_{k=1}^{K} (-1)^k \sum_{C \in Gph_i} N(k, C) \cdot |\Delta_C(G^{I_C}, \bar{b})|,$$

where $N(k, C)$ counts the number of unordered sets of embeddings $\{\iota_1, \ldots, \iota_k\}$ of graphs in $\{A_1, \ldots, A_l\}$ into C such that C is the union of their images.

- To find the cardinality of $\{a : \bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z}) \wedge \bigwedge_j \neg \exists \bar{z} \Delta_{D_j}(a, \bar{b}, \bar{z})\}$, we appeal to inclusion-exclusion again, as

$$|A \cap \bigcap_{j=1}^{l} (A \setminus B_j)| = |A| - \sum_k \sum_{i_1 < \ldots < i_k \leq l} |A \cap \bigcap_j B_{i_j}|.$$

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- This expression is a $\mathbb{Z}$-linear combination of numbers asymptotically of the form $n^{a-b\alpha}$. The two flawed arguments from the quantifier-free case apply again, and the fix is similar though more complicated.

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- Briefly: we only care about counting graph extensions of $\bar{b}$ to finitely many graphs C. Consider the rigid parts R_C of each such extension. We can definably specify, for $\bar{b}$, the number of copies to each R_C, and the ways in which these copies intersect.

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- For every possible such specification, we show, as in the quantifier-free case, that when leading terms in the associated "polynomial" ($\sum kn^{\gamma}$ for $\gamma \in \mathbb{R}$) cancel, they cancel in their entirety, and the resulting combination is still asymptotically equal to such a polynomial.

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