Probabilistic $R$-Macs

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“Asymptotic class” is a family of concepts about finite structures which have restrictions on the cardinalities of definable sets. The first to be explicitly named was the one-dimensional asymptotic classes:

Definition (Macpherson & Steinhorn 08)
A family \((M_k : k \in \omega)\) of finite \(L\)-structures is a one-dimensional asymptotic class if for every \(L\)-formula \(\phi(x_1, \ldots, x_n, \bar{y})\), there are \(L\)-formulas \(\pi_1(\bar{y}), \ldots, \pi_r(\bar{y})\), pairs \((\mu_1, d_1), \ldots, (\mu_r, d_r)\) \(\in \mathbb{R} \times \{1, \ldots, n\}\) and a number \(C \in \mathbb{R}\) such that the sets \(\pi_1(M|\bar{y}|k), \ldots, \pi_r(M|\bar{y}|k)\) partition \(M|\bar{y}|k\) for each (sufficiently large) \(k\), and if \(\bar{b} \in \pi_i(M|\bar{y}|k)\) then 
\[|\phi(M^n k, \bar{b}) - \mu_i|_{M k} < C|M|d_i - 1/2.\]

Theorem (Chatzidakis, Van den Dries & Macintyre 92)
The class of finite fields is a one-dimensional asymptotic class.
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A family \((M_k : k \in \omega)\) of finite \(L\)-structures is a **one-dimensional asymptotic class** if for every \(L\)-formula \(\varphi(x_1, \ldots, x_n, \bar{y})\), there are \(L\)-formulas \(\pi_1(\bar{y}), \ldots, \pi_r(\bar{y})\), pairs \((\mu_1, d_1), \ldots, (\mu_r, d_r) \in \mathbb{R} \times \{1, \ldots, n\}\) and a number \(C \in \mathbb{R}\) such that
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- the sets \(\pi_1(M_k|\bar{y}|), \ldots, \pi_r(M_k|\bar{y}|)\) partition \(M_k|\bar{y}|\) for each (sufficiently large) \(k\), and

\[|\varphi(M_k^n, \bar{b}) - \mu_i|_{M_k|\bar{y}|} < C\]
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- if \(\bar{b} \in \pi_i(M_k^{\bar{y}})\) then \(|\varphi(M_k^n, \bar{b}) - \mu_i M_k^{d_i}| < C |M|^{d_i-1/2}\).
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**Theorem (Chatzidakis, Van den Dries & Macintyre 92)**

The class of finite fields is a one-dimensional asymptotic class.
Definition (Anscombe, Macpherson, Steinhorn, Wolf 16)

Let $R$ be a set of functions $h : \{M : M$ is an $L$-structure$\} \rightarrow \mathbb{R}$. A sequence $(M_k : k \in \omega)$ of $L$-structures is an $R$-multidimensional asymptotic class ($R$-mac) if for every $L$-formula $\varphi(\bar{x}, \bar{y})$, there are $L$-formulas $\pi_1(\bar{y}), \ldots , \pi_r(\bar{y})$ and functions $h_1, \ldots , h_r \in R$ such that
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A one-dimensional asymptotic class is an $R$-mac where $R$ is the set of functions $\{ M \mapsto \mu | M | : \mu \in R, d \in \mathbb{N} \}$.

An $N$-dimensional asymptotic class (introduced by Elwes) is an $R$-mac where $R$ is the set of functions $\{ M \mapsto \mu | M |^{d/N} : \mu \in R, d \in \mathbb{N} \}$.
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An $N$-dimensional asymptotic class (introduced by Elwes) is an $R$-mac where $R$ is the set of functions $\{ M \mapsto \mu|M|^{d/N} : \mu \in \mathbb{R}, d \in \mathbb{N} \}$.
Definition

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In this talk, all random structures considered will be finite – so there is no care we need to take with measurability issues, and we may consider the probability of any property of the random structure $\hat{M}$. 
Let $n \in \omega$ and $p \in [0, 1]$. The Erdős-Rényi random graph $\hat{G}(n, p)$ is the random graph on $n$ vertices formed by letting each of the $\binom{n}{2}$ possible edges appear independently with probability $p$. 

Definition

Let $\alpha \in (0, 1)$. The Spencer-Shelah random graph sequence with parameter $\alpha$ is the sequence $(\hat{G}_\alpha(n) : n \in \omega)$. 

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Definition

Let \( \alpha \in (0, 1) \). The \textit{Spencer-Shelah random graph sequence with parameter} \( \alpha \) is the sequence \( (\hat{G}_n^\alpha := \hat{G}(n, n^{-\alpha}) : n \in \omega) \).
**Definition**

Let \((M_n : n \in \omega)\) be a sequence of random \(L\)-structures. The **zero-one theory** of the sequence is the set of \(L\)-sentences

\[
\{ \phi : \lim_{n \to \infty} \mathbb{P}(M_n \models \phi) = 1 \}.
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Theorem (Fagin)

For a fixed $p \in (0, 1)$, the sequence $(\hat{G}(n, p) : n \in \omega)$ has the theory of the Rado random graph as its zero-one theory.
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Theorem (Spencer & Shelah)

If \(\alpha\) is irrational, the Spencer-Shelah random graph sequence with parameter \(\alpha\) has a complete zero-one theory \(T_\alpha\).
Definition (V.)

Let $R$ be a set of functions from $\{M : M$ is an $L$-structure$\}$ to $\mathbb{R}$. Let $\varphi(\bar{x}; \bar{y})$ be an $L$-formula.

A sequence $(\hat{M}_n : n \in \omega)$ of random $L$-structures is a probabilistic $R$-mac for $\varphi$ if there are $L$-formulas $\pi_1(\bar{y}), \ldots, \pi_r(\bar{y})$ and functions $h_1, \ldots, h_r \in R$ such that for every $\epsilon > 0$, the probabilities of the following statements go to 1 as $n$ goes to infinity:

- the sets $\pi_1(M_n\mid\bar{y}\mid), \ldots, \pi_r(M_n\mid\bar{y}\mid)$ partition $\hat{M}_n\mid\bar{y}\mid$, and
- if $\bar{b} \in \pi_i(\hat{M}_n\mid\bar{y}\mid)$ then $(1 - \epsilon)h_i(\hat{M}_n) < |\varphi(\hat{M}_n\mid\bar{x}\mid, \bar{b})| < (1 + \epsilon)h_i(\hat{M}_n)

$(\hat{M}_n : n \in \omega)$ is a probabilistic $R$-mac if it is a probabilistic $R$-mac for every formula $\varphi(\bar{x}, \bar{y})$. 

Note: The functions $h_i$ are defined on (deterministic) $L$-structures; the expression $h_i(\hat{M}_n)$ is a real-valued random variable.
Definition (V.)

Let $R$ be a set of functions from $\{ M : M \text{ is an } L\text{-structure}\}$ to $\mathbb{R}$. Let $\varphi(\bar{x}; \bar{y})$ be an $L$-formula.

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Proposition

Let \((\hat{M}_n : n \in \omega)\) be a probabilistic R-mac. For each \(n\), let \(X_{n,1}, X_{n,2}, \ldots\) be a sequence of statements (events) about the structure \(\hat{M}_n\). Suppose that for each \(k\), we have

\[
\liminf_{n \to \infty} \mathbb{P}(\hat{M}_n \text{ satisfies } X_{n,1}, X_{n,2}, \ldots, \text{ and } X_{n,k}) > 0.
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Then we can find an R-mac \((M_1, M_2, \ldots)\) such that for every \(k\), \(M_n\) satisfies \(X_{n,k}\) for cofinitely many \(n\).
Proposition

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$$\lim_{n \to \infty} \inf \mathbb{P}(\hat{M}_n \text{ satisfies } X_{n,1}, X_{n,2}, \ldots, \text{ and } X_{n,k}) > 0.$$ 

Then we can find an R-mac $(M_1, M_2, \ldots)$ such that for every $k$, $M_n$ satisfies $X_{n,k}$ for cofinitely many $n$.

Note

For example, we can ensure that the zero-one theory of the $\hat{M}_n$ is (contained in) the limit theory of the $M_n$. 
Proposition

To show that a sequence \((\hat{M}_n : n \in \omega)\) of random structures is a probabilistic \(R\)-mac, it suffices to show that it is a probabilistic \(R\)-mac for every formula \(\varphi(x_1; \bar{y})\) with a single object variable (provided that \(R\) is asymptotically closed under addition and multiplication).

Proof. Routine fiber-decomposition proof. To show \(R\) is a probabilistic \(R\)-mac for \(\varphi(x_1; x_2. \ldots x_k; \bar{y})\), re-contextualize the variables as \(\varphi(x_1; x_2. \ldots x_k; \bar{y})\) and obtain estimates for \(|\varphi(\hat{M}_n; a_2 \ldots a_k; \bar{b})|\), definably for \(a_2 \ldots a_k; \bar{b}\) by formulas \(\pi_i(x_2. \ldots x_k; \bar{y})\). By induction, obtain cardinality bounds for \(|\pi_i(\hat{M}_{n-1}; \bar{b})|\), and use these to bound the size of \(|\varphi(\hat{M}_n; \bar{b})|\).
**Proposition**

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**Proof.**

Routine fiber-decomposition proof. To show $R$ is a probabilistic $R$-mac for $\varphi(x_1 \ldots x_k; \bar{y})$, re-contextualize the variables as $\varphi(x_1; x_2 \ldots x_k \bar{y})$ and obtain estimates for $|\varphi(\hat{M}_n; a_2 \ldots a_k \bar{b})|$, definably for $a_2 \ldots a_k \bar{b})$ by formulas $\pi_i(x_2 \ldots x_k \bar{y})$. By induction, obtain cardinality bounds for $|\pi_i(\hat{M}_n^{k-1}; \bar{b})|$, and use these to bound the size of $|\varphi(\hat{M}_n^k, \bar{b})|$.
If $\mathcal{C}$ is a class of (deterministic) finite structures, we say $\mathcal{C}$ is a probabilistic $R$-mac if $(\hat{M}_n : n \in \omega)$ is, where $\hat{M}_n$ is the uniform distribution on all structures in $\mathcal{C}$ with universe $\{1, \ldots, n\}$. 

Proposition

The class of all finite graphs is a probabilistic $R$-mac, where $R$ is the set $\{M_7 \mapsto \mu | |M| \leq d : \mu \in \mathbb{R}, d \in \mathbb{N}\}$ (a probabilistic one-dimensional asymptotic class).
If \( C \) is a class of (deterministic) finite structures, we say \( C \) is a probabilistic \( R \)-mac if \((\hat{M}_n : n \in \omega)\) is, where \( \hat{M}_n \) is the uniform distribution on all structures in \( C \) with universe \( \{1, \ldots, n\} \).

**Proposition**

The class of all finite graphs is a probabilistic \( R \)-mac, where \( R \) is the set \( \{M \mapsto \mu |M|^d : \mu \in \mathbb{R}, d \in \mathbb{N}\} \) (a probabilistic one-dimensional asymptotic class).
Proof Sketch

- The sequence of uniform distributions for finite graphs is the same as the Erdős-Rényi sequence \( \hat{G}(n, 1/2) : n \in \omega \).
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- By quantifier elimination and the one-variable lemma, it suffices to check the $R$-pmac condition for formulas of the form $\phi(x, \bar{y}) = \bigwedge_i x E y_i \land \bigwedge_j \neg x E y_j \land \rho(x, \bar{y})$, where $\rho(x, \bar{y})$ expresses that all elements are distinct.

That is, we wish to obtain estimates for $|\{a \in \hat{G}_n : \hat{G}_n | = \phi(a, \bar{b})\}|$ for $\bar{b} \in \hat{G}_n$. Assuming that all $b_i$ are distinct, the probability that a given $a/\in \bar{b}$ satisfies $\phi(a, \bar{b})$ is $2^{-|\bar{y}|}$. Furthermore, these events are mutually independent for different $a$. 

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- That is, we wish to obtain estimates for $|\{a \in \hat{G}_n : \hat{G}_n \models \phi(a, \bar{b})\}|$ for $\bar{b} \in \hat{G}_n$.

- Assuming that all $b_i$ are distinct, the probability that a given $a \notin \bar{b}$ satisfies $\phi(a, \bar{b})$ is $2^{-|\bar{y}|}$. Furthermore, these events are mutually independent for different $a$. 
Hoeffding’s Inequality (Bernoulli Version)

Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables. Let $S_n = \sum_{i \leq n} X_i$. Then $\mathbb{P}(|S_n - \mathbb{E}[S_n]| \geq t) \leq e^{-2t^2/n}$.
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Proof Sketch Cont’d

- $|\{a \in \hat{G}_n : a\bar{b} \in \phi(\hat{G}_n^{1+|\bar{y}|})\}|$ is a sum of Bernoulli RV’s as in Hoeffding’s Inequality, with expectation $(n - |\bar{y}|) \cdot 2^{-k} = 2^{-k} n + c$
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- For a given $\epsilon$, the probability that this number is not between $(1 - \epsilon)2^{-k} n$ and $(1 + \epsilon)2^{-k} n$ is $\leq \exp(-2(\epsilon 2^{-k} n)^2/n) = \exp(-\eta n)$ for $\eta = 2^{-2k-1} \epsilon$. This goes to 0 as $n \to \infty$. 
Proposition

The class of all finite partial orders is a probabilistic one-dimensional asymptotic class.
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Proof Sketch

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- We rely on the following theorem of Kleitman & Rothschild (75):
- Asymptotically most finite partial orders on \( \{1, \ldots, n\} \) have three layers in their Hasse diagram, with middle layer of size approximately \( n/2 \) and top and bottom layers approximately \( n/4 \).
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  - The uniform distribution on these partial orders can be modeled as an independent-coin-flip random structure, similar to the random graph.
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- Therefore single-variable definable sets in the random partial order on \{1, \ldots, n\} have cardinality approximately \(2^{-k}n\) for some \(k\), by the same argument as in the random graph.
Theorem

For irrational $\alpha \in (0, 1)$, the Spencer-Shelah random graph sequence is a probabilistic $R^\alpha$-mac, where $R^\alpha$ is the set of functions
\[
\{ M \mapsto k|M|^{a-b\alpha} : k, a, b \in \mathbb{N} \}.
\]
Theorem

For irrational $\alpha \in (0, 1)$, the Spencer-Shelah random graph sequence is a probabilistic $R^\alpha$-mac, where $R^\alpha$ is the set of functions

$\{M \mapsto k |M|^{a-b\alpha} : k, a, b \in \mathbb{N}\}$.

Proof Sketch

- We rely on the following quantifier elimination result from Laskowski.

Theorem (Laskowski 07)
The theory $T_\alpha$ admits quantifier elimination down to formulas of the form $\phi(\bar{x}) := \exists \bar{z} \Delta_B(\bar{x} \bar{z})$.

Proof Sketch Continued

Observe: complete quantifier-free formulas are formulas of the above form for $|\bar{z}| = 0$. 
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- Notation: For a graph $B$ on $\{1, \ldots, k\}$, let $\Delta_B(v_1 \ldots v_k)$ be the formula $\bigwedge_{i \neq j} v_i E v_j \land \bigwedge_{i \neq j} \neg v_i E v_j$. Let $\Delta^+_B(v_1 \ldots v_k)$ be $\bigwedge_{i \neq j} v_i E v_j$. 
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Proof Sketch

- We rely on the following quantifier elimination result from Laskowski.
- Notation: For a graph $B$ on $\{1, \ldots, k\}$, let $\Delta_B(v_1 \ldots v_k)$ be the formula $\bigwedge_{i \in E_j} v_i E v_j \land \bigwedge_{i \notin E_j} \neg v_i E v_j$. Let $\Delta^+_B(v_1 \ldots v_k) = \bigwedge_{i \in E_j} v_i E v_j$.

Theorem (Laskowski 07)

The theory $T^\alpha$ admits quantifier elimination down to formulas of the form $\phi(x) := \exists z \Delta_B(xz)$.

Proof Sketch Continued

- Observe: complete quantifier-free formulas are formulas of the above form for $|z| = 0$. 
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For irrational $\alpha \in (0, 1)$, the Spencer-Shelah random graph sequence is a probabilistic $R^\alpha$-mac, where $R^\alpha$ is the set of functions

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Proof Sketch Continued

- By quantifier elimination, it suffices to show that $(\hat{G}_n^\alpha : n \in \omega)$ is an $R^\alpha$-pmac for conjunctions of formulas of the form

  $$\phi(x, \bar{y}) := \exists \bar{z} \Delta_B(x, \bar{y}, \bar{z})$$

  and their negations.
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- Given \( \bar{b} \in \hat{G}_n^\alpha \), we show how to write the cardinality of
\[
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as a linear combination (possibly with negative coefficients) of cardinalities \(|\{\bar{c} : \Delta_C(\bar{b}, \bar{c})\}|\) as \( C \) ranges over finitely many graphs.
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- By quantifier elimination, it suffices to show that $(\hat{G}_n^\alpha : n \in \omega)$ is an $R^\alpha$-pmac for conjunctions of formulas of the form
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- We first show asymptoticity results for quantifier-free formulas (i.e. the case $|\bar{z}| = 0$)

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- We then sketch how to deal with including conjuncts of the form $\neg \exists \bar{z} \Delta_B(x, \bar{b}, \bar{z})$. 
Definition

Let $A \subseteq B$ be graphs. $\delta_\alpha(B/A)$ is the quantity $\nu - e_\alpha$, where $\nu$ is the number of vertices in $B \setminus A$ and $e$ is the number of edges in $B$ which do not have both endpoints in $A$. 

We say that $B$ is safe over $A$ if for every $A \subset C \subseteq B \subseteq A$, $\delta(C/A) > 0$. We say that $B$ is rigid over $A$ if for every $A \subseteq C \subset B$, $\delta(B/C) < 0$. 

$T_\alpha$ implies the following sentences about the graph $G$:

1. If $\delta(A/\emptyset) < 0$, then $G$ contains no copy of $A$.
2. If $A \subseteq B$ is a safe extension, then every copy of $A$ extends to a copy of $B$.
3. If $A \subseteq B$ is a rigid extension, then every copy of $A$ extends to at most $K(B/A)$ copies of $B$ (for some fixed number $K(B/A)$).
Definition
Let $A \subseteq B$ be graphs. $\delta_\alpha(B/A)$ is the quantity $v - e\alpha$, where $v$ is the number of vertices in $B \setminus A$ and $e$ is the number of edges in $B$ which do not have both endpoints in $A$.

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Proposition (Spencer & Shelah)

Let $B$ be a graph on $\{1, \ldots, k, k + 1, \ldots, k + l\}$, and let $A$ be the subgraph on $\{k + 1, \ldots, k + l\}$. Then for any $\bar{b} \in \Delta^+(G^l_n)$ ($\bar{b} \in \Delta(G^l_n)$),

\[ \mathbb{E}[|\Delta^+_B(G^k_n, \bar{b})|] \approx n^{\delta(B/A)} \approx \mathbb{E}[|\Delta_B(G^k_n, \bar{b})|]. \]

Proof.
There are \((n-l)(n-l-1)\ldots(n-l-(k-1))\) \(\approx n^{k}\) extensions of $\bar{b}$ to \((k+l)\)-tuples $\bar{a}\bar{b}$. For any such \((k+l)\)-tuple to be an element of $\Delta^+_B(G^k_n, \bar{b})$, we need each of the $e$ edges of $B$ to occur, where $e$ is the number in $\delta(B/A) = v-e\alpha$. The probability that this happens is \((n-\alpha)e = n-e\alpha\). By linearity of expectation, the expected number of elements in $\Delta^+_B(G^k_n, \bar{b})$ is \(\approx n^{k}\cdot(n-\alpha) = n^{k} - e\alpha = n^{\delta(B/A)}\). For $\delta_B(G^k_n, \bar{b})$, we note that $\bar{a} \in \delta_B(G^k_n, \bar{b})$ iff $\bar{a} \in \Delta^+_B(G^k_n, \bar{b})$ and $\bar{a}/\in \Delta^+_B(G^k_n, \bar{b})$ for any graph $B'$ for which $B$ is a proper spanning subgraph, and for any such $B'$, $\delta(B'/A) \leq \delta(B/A) - \alpha$. 


Proposition (Spencer & Shelah)

Let $B$ be a graph on $\{1, \ldots, k, k+1, \ldots, k+l\}$, and let $A$ be the subgraph on $\{k+1, \ldots, k+l\}$. Then for any $\bar{b} \in \Delta^+(G_n^l)$ ($\bar{b} \in \Delta(G_n^l)$),

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There are $(n-l)(n-l-1) \ldots (n-l-(k-1)) \approx n^k$ extensions of $\bar{b}$ to $(k+l)$-tuples $\bar{a}\bar{b}$. For any such $(k+l)$-tuple to be an element of $\Delta^+_B(G_n^k, \bar{b})$, we need each of the $e$ edges of $B$ to occur, where $e$ is the number in $\delta(B/A) = n-e\alpha$. The probability that this happens is $(n^{-\alpha})^e = n^{-e\alpha}$. By linearity of expectation, the expected number of elements in $\Delta^+(G_n^l)$ is $\approx n^k \cdot n^{-e\alpha} = n^{k-e\alpha} = n^{\delta(B/A)}$.

For $\delta_B(G_n^k, \bar{b})$, we note that $\bar{a} \in \delta_B(G_n^k, \bar{b})$ iff $\bar{a} \in \Delta^+_B(G_n^k, \bar{b})$ and $\bar{a} \notin \Delta^+_B(G_n^k, \bar{b})$ for any graph $B'$ for which $B$ is a proper spanning subgraph, and for any such $B'$, $\delta(B'/A) \leq \delta(B/A) - \alpha$. \qed
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\[
\mathbb{E}[|\Delta^+_B(G^k_n, \bar{b})|] \approx n^{\delta(B/A)} \left( \approx \mathbb{E}[|\Delta_B(G^k_n, \bar{b})|] \right).
\]

However, this is different from saying that the actual number of extensions of $\bar{b}$ is asymptotically equal to $n^{\delta(B/A)}$. This is not generally the case: if $B$ is not safe over $A$, then most copies of $A$ do not extend to $B$, even though the expected number of copies is positive.
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On the other hand...
**Theorem (Kim & Vu (2000))**

Let $B$ be a graph on $\{1, \ldots, k, k+1, \ldots, k+l\}$ and let $A$ be the subgraph on $\{k+1, \ldots, k+l\}$. Suppose that $B$ is safe over $A$. Then there is a positive constant $\epsilon > 0$ such that the probability of the statement

$$\text{for all } \vec{b} \in \Delta_A^+(G_n), n^\delta(B/A) \cdot (1 - n^{-\epsilon}) < |\Delta_B^+(G_n^k, \vec{b})| < n^\delta(B/A) \cdot (1 + n^{-\epsilon})$$

goes to 1 as $n$ goes to $\infty$. 

**Corollary**

$\hat{G}^\alpha_n$ is a probabilistic $R^\alpha$-mac for the formula $\Delta_B^+(\vec{x}; \vec{y})$, with measuring functions $G \to 0$ and $h: G \to |G|_{B/A}$. 

**Proof.** For every $\epsilon$, with probability approaching 1: if $\hat{G}^\alpha_n | = \Delta_A^+(G_n^l)$ then 

$$(1 - \epsilon) h(\hat{G}^\alpha_n) < |\Delta_B^+(G_n^k, \vec{b})| < (1 + \epsilon) h(\hat{G}^\alpha_n).$$

If $\hat{G}^\alpha_n \not| = \Delta_A^+(G_n^l)$ then $|\Delta_B^+(G_n^k, \vec{b})| = 0$. 

Alex Van Abel (Wesleyan University) 

Probabilistic $R$-Macs

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for all $\bar{b} \in \Delta^+_A(G^n), \ n^{\delta(B/A)} \cdot (1 - n^{-\epsilon}) < |\Delta^+_B(G^n_k, \bar{b})| < n^{\delta(B/A)} \cdot (1 + n^{-\epsilon})$

goes to 1 as $n$ goes to $\infty$.

Corollary

$(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic $R^\alpha$-mac for the formula $\Delta^+_B(\bar{x}; \bar{y})$, with measuring functions $G \mapsto 0$ and $h : G \mapsto |G|^{B/A}$.
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\text{for all } \bar{b} \in \Delta^+_A(G_n^l), n^{\delta(B/A)} \cdot (1 - n^{-\epsilon}) < |\Delta^+_B(G_n^k, \bar{b})| < n^{\delta(B/A)} \cdot (1 + n^{-\epsilon})
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Proof.

For every $\epsilon$, with probability approaching 1: if $\hat{G}_n \models \Delta^+_A(\bar{b})$ then

$$(1 - \epsilon)h(\hat{G}_n^\alpha) < |\Delta_B((\hat{G}_n^\alpha)^{|\bar{x}|}, \bar{b})| < (1 + \epsilon)h(\hat{G}_n^\alpha).$$

If $\hat{G}_n \not\models \Delta^+_A(\bar{b})$ then

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**Proposition**

Let $A \subseteq B$ be a graph extension. Then there is a unique intermediate subgraph $A \subseteq rs(A, B) \subseteq B$ such that $rs(A, B)$ is rigid over $A$ and $B$ is safe over $rs(A, B)$. 

**Definition**

Define $\delta^\star(B/A)$ to be $\delta(B/R)$, where $R = rs(A, B)$. Define $K^\star(B/A)$ to be $K(R/A)$ (the maximum number of extensions of a copy of $A$ to a copy of $R$ in $T_\alpha$).

**Corollary**

Let $B$ be a graph. Then $(\hat{G}_\alpha^n : n \in \omega)$ is a probabilistic $R_\alpha$-mac for $\Delta^+ B(\bar{x}, \bar{y})$. 
It is relatively easy to remove the restriction that $B$ is safe.

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**Corollary**

Let $B$ be a graph. Then $(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic $R^\alpha$-mac for $\Delta_B^+(\bar{x}, \bar{y})$. 
Corollary

Let $B$ be a graph on $\{1, \ldots, k + l\}$. Let $A$ be the induced subgraph on $\{k + 1, \ldots, k + l\}$. Then $(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic $R^\alpha$-mac for $\Delta^+_B(x_1 \ldots x_k ; y_1 \ldots y_l)$, with measuring functions $G \mapsto m|G|^{\delta^*(B/A)}$ with $m \in \{0, 1, \ldots, K^*(B/A)\}$.
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Proof.

- We have $A \subseteq R \subseteq B$ with $R$ rigid over $A$ and $B$ safe over $R$. 
Corollary

Let \( B \) be a graph on \( \{1, \ldots, k + l\} \). Let \( A \) be the induced subgraph on \( \{k + 1, \ldots, k + l\} \). Then \((\hat{G}_n^\alpha : n \in \omega)\) is a probabilistic \( R^\alpha \)-mac for \( \Delta_B^+(x_1 \ldots x_k; y_1 \ldots y_l) \), with measuring functions \( G \mapsto m|G|^{\delta^*(B/A)} \) with \( m \in \{0, 1, \ldots, K^*(B/A)\} \).

Proof.

- We have \( A \subseteq R \subseteq B \) with \( R \) rigid over \( A \) and \( B \) safe over \( R \).
- If \( \bar{b} \nmid \Delta_A^+(\bar{y}) \) then \( |\Delta_B^+(\hat{G}_n^{\bar{x}}, \bar{b})| = 0 \).
Corollary

Let $B$ be a graph on \{1, \ldots, k + l\}. Let $A$ be the induced subgraph on \{k + 1, \ldots, k + l\}. Then $(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic $R^\alpha$-mac for $\Delta_+^B(x_1 \ldots x_k; y_1 \ldots y_l)$, with measuring functions $G \mapsto m |G|^{\delta^*(B/A)}$ with $m \in \{0, 1, \ldots, K^*(B/A)\}$.

Proof.

- We have $A \subseteq R \subseteq B$ with $R$ rigid over $A$ and $B$ safe over $R$.
- If $\not= \Delta_+^A(\bar{y})$ then $|\Delta_+^B(\hat{G}_n^{|\bar{x}}), \bar{b})| = 0$.
- If $\models \Delta_+^A(\bar{y})$ we first count the number of extensions of $\bar{b}$ to a copy of $R$ – with probability approaching 1, there are $\leq K(R/A)$ such extensions for any such $\bar{b}$. Note that “$\bar{b}$ has $m$ extensions to $R$” is definable.
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Let $B$ be a graph on $\{1, \ldots, k + l\}$. Let $A$ be the induced subgraph on $\{k + 1, \ldots, k + l\}$. Then $(\hat{G}_n^\alpha : n \in \omega)$ is a probabilistic $R^\alpha$-mac for $\Delta_B^+ (x_1 \ldots x_k ; y_1 \ldots y_l)$, with measuring functions $G \mapsto m|G|^{\delta^*(B/A)}$ with $m \in \{0, 1, \ldots, K^*(B/A)\}$.

Proof.

- We have $A \subseteq R \subseteq B$ with $R$ rigid over $A$ and $B$ safe over $R$.
- If $\bar{b} \not\models \Delta_A^+ (\bar{y})$ then $|\Delta_B^+ (\hat{G}_n^{\bar{x}}, \bar{b})| = 0$.
- If $\bar{b} \models \Delta_A^+ (\bar{y})$ we first count the number of extensions of $\bar{b}$ to a copy of $R$ – with probability approaching 1, there are $\leq K(R/A)$ such extensions for any such $\bar{b}$. Note that “$\bar{b}$ has $m$ extensions to $R$” is definable.
- If $\bar{b}$ has $m$ extensions to $R$, then each of those extensions has approximately $n^{\delta(B/R)}$ extensions to $B$, so $\bar{b}$ has in total approximately $mn^{\delta(B/R)}$ extensions to $B$. That is, $|\Delta_A^+ (\hat{G}_n^{\bar{x}}, \bar{b})|$ is approximately $mn^{\delta(B/R)}$. 
Now we go from $\Delta_B^+(\bar{x}\bar{y})$ to $\Delta_B(\bar{x}\bar{y})$. 
Now we go from $\Delta^+_B(\overline{x}\overline{y})$ to $\Delta_B(\overline{x}\overline{y})$.

Observe that $\Delta_B(\overline{x}\overline{y})$ is equivalent to $\Delta^+_B(\overline{x}\overline{y}) \land \bigwedge_{B'} \neg \Delta^+_{B'}(\overline{x}\overline{y})$, where $B'$ ranges over all graphs $B'$ obtained by exactly one edge to $B$. By inclusion-exclusion, we obtain

$$|\Delta_B(\overline{x}, \overline{b})| = |\Delta^+_B(\overline{x}, \overline{b})| - \sum_k (-1)^k \sum_{B' \in B(k)} |\Delta^+_{B'}(\overline{x}, \overline{b})|,$$

where $B(k)$ is the collection of graphs obtained by adding $k$ edges to $B$. Two well-intentioned but incorrect arguments:

1. This sum is a linear combination of terms each approximately of the form $kn\gamma$ for some $k \in \mathbb{Z}$ and $\gamma \in \mathbb{R}$. Such a "polynomial" is approximately equal to its leading term.
2. This sum is asymptotically equivalent to $|\Delta^+_B(\overline{x}, \overline{b})|$—edges are rare, and so almost all positive copies of $B$ should be full copies of $B$ (i.e., $B$ as an induced subgraph) — that is, $|\Delta^+_{B'}(\overline{x}, \overline{b})| = o(|\Delta^+_B(\overline{x}, \overline{b})|)$. 
Now we go from $\Delta_B^+(\bar{x}\bar{y})$ to $\Delta_B(\bar{x}\bar{y})$.

Observe that $\Delta_B(\bar{x}\bar{y})$ is equivalent to $\Delta_B^+(\bar{x}\bar{y}) \land \bigwedge_{B'} \neg \Delta_{B'}^+(\bar{x}\bar{y})$, where $B'$ ranges over all graphs $B'$ obtained by exactly one edge to $B$. By inclusion-exclusion, we obtain

$$|\Delta_B(\bar{x}, \bar{b})| = |\Delta_B^+(\bar{x}, \bar{b})| - \sum_k (-1)^k \sum_{B' \in B(k)} |\Delta_{B'}^+(\bar{x}, \bar{b})|,$$

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Sketch of problem and solution: if we only add edges to the rigid part $R$ of $B$ over $A$, it is possible to have $\delta(B'/R) = \delta(B/R)$. In this case, every extension of $R'$ (the rigid part of $B'$) to $B'$ is already an extension of $R$ to $B$ (since we are adding no new edges from $R$ to the safe part of $B$). Therefore in the inclusion-exclusion formula, when we subtract this number of extensions, we are cancelling out an earlier term in its entirety, not asymptotically as in the earlier bullet point.

(Board example)
We now show how to find the cardinality of \( \{ a : \bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z}) \} \).

Note: by the theory \( T_\alpha \), it suffices to consider the case where each \( B_i \) is rigid over the graph on \( a, \bar{b} \).

Given \( B_1, \ldots, B_k \) as above, we can find \( A_1, \ldots, A_l \) such that for any \( a, \bar{b}, \bar{z} \), \( \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z}) \) if and only if \( \exists \bar{z} \Delta_{A_j}(a, \bar{b}, \bar{z}) \). (Picture proof on board)

For a given \( \bar{b} \), we wish to count the number of \( a \) such that \( \hat{G}_{\alpha | n} = \exists \bar{z} \Delta_{A_j}(a, \bar{b}, \bar{z}) \).

To do this, we enumerate every instance of \( a \) occurring in a copy of some \( A_j \) over \( \bar{b} \).

By inclusion-exclusion principle, we add up all instances of \( a \) which occur at least once, subtract the number of instances of \( a \) occurring twice, add the number of instances of \( a \) occurring three times...
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Instances of $a$ occurring at least once are given by the total number of extensions of $\bar{b}$ to some $A_i$. 

By rigidity, there is some $K$ such that no $a\bar{b}$ extends to a copy of $C$ where $C$ can be written as a union of $\geq K$ instances of $A_i$’s – since the number of copies of $a\bar{b}$ to any one $A_i$ is (with probability $\omega$) bounded by some $K_i$. So this process terminates!

We are left with

$$\{a : \exists \bar{z} \Delta B_i (a, \bar{b}, \bar{z}) \} = \sum_{k=1}^{K} \left( -\frac{1}{2} \right)^{k} \times \sum_{C \in \text{Gph}_i} N(k, C) \times |\Delta C(G, \bar{b})|,$$

where $N(k, C)$ counts the number of unordered sets of embeddings $\{\iota_1, \ldots, \iota_k\}$ of graphs in $\{A_1, \ldots, A_l\}$ into $C$ such that $C$ is the union of their images.
Instances of $a$ occurring at least once are given by the total number of extensions of $\bar{b}$ to some $A_i$.

Instances of $a$ occurring at least \textit{twice} are given by counting extensions of $\bar{b}$ to graphs $C$ which can be written as a union as $A_i \cup A_j$ over $a\bar{b}$, counting once for every such union. And so on.
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\{ a : \bigwedge_i \exists \bar{z} \Delta_{B_i}(a, \bar{b}, \bar{z}) \land \bigwedge_j \neg \exists \bar{z} \Delta_{D_j}(a, \bar{b}, \bar{z}) \}, \) we appeal to inclusion-exclusion again, as

\[
|A \cap \bigcap_{j=1}^{l} (A \setminus B_j)| = |A| - \sum_{k} \sum_{i_1 < \ldots < i_k \leq l} |A \cap \bigcap_{j} B_{i_j}|.
\]

This expression is a \( Z \)-linear combination of numbers asymptotically of the form \( n^{a-b^{\alpha}} \). The two flawed arguments from the quantifier-free case apply again, and the fix is similar though more complicated. Briefly: we only care about counting graph extensions of \( \bar{b} \) to finitely many graphs \( C \). Consider the rigid parts \( R_C \) of each such extension. We can definably specify, for \( \bar{b} \), the number of copies to each \( R_C \), and the ways in which these copies intersect. For every possible such specification, we show, as in the quantifier-free case, that when leading terms in the associated "polynomial" (\( P_{kn}^\gamma \) for \( \gamma \in R \)) cancel, they cancel in their entirety, and the resulting combination is still asymptotically equal to such a polynomial.
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References